THE THEORY OF DETERMINANTS

IN THE

HISTORICAL ORDER OF ITS DEVELOPMENT

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THE THEORY OF DETERMINANTS IN THE HISTORICAL ORDER OF ITS DEVELOPMENT

PART I.

DETERMINANTS IN GENERAL

LEIBNITZ (1693) TO CAYLEY (1841)

BY

THOMAS MUIR, M.A., LL.D., F.R.S.E.

AUTHOR OF "A TREATISE ON THE THEORY OF DETERMINANTS," AND OTHER WORKS

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PREFACE.

DURING the writing of my *Treatise on the Theory of Determinants* (Macmillan & Co., 1882), it was repeatedly forced on my attention that the history of the subject had been very imperfectly looked into. Not only, as it appeared, had injustice been done by the attribution of isolated theorems and demonstrations to authors other than the first discoverers, but the labours of the great founders of the theory had been disproportionately represented, and a considerable amount of valuable work had actually been lost sight of altogether. This state of matters, it was clear, had its explanation in the fact that two of the foremost nationalities of Europe had not contributed their proper shares to the work of historical research. France, for some reason or other, had taken comparatively slight interest in any part of the subject; and England, though interested in the *theory* and contributing largely to its development, had been content as regards the *history* simply to accept and promulgate the results of German assiduity.* Not unnaturally, therefore, mathematicians in general had come to look at the history almost as it were through German spectacles. It could not be said, moreover, that this optical help was detri-

* The apathy of France seems the more blameworthy, because, as will appear, the theory in its early stages was essentially a creation of French genius. As for England, signs have not been awanting during recent years that she means for the future to take her part in the investigation and exposition of mathematical history.

mental to authors outside Germany and to them alone, for several German mathematicians of note had not had justice done them, their writings * in one or two instances being even unknown in the land that produced them.

With these facts before me, I resolved to set about collecting the whole literature of the subject, in order that, as a first step on the way to a history, a *bibliography* might be compiled. To those who know how lamentably ill-provided mathematicians are with guides to their literature, it will readily occur that this initiatory step entailed a vast amount of labour. When so far completed, the "List of Writings," as it was called, was published in the *Quarterly Journal of Mathematics*, of which it occupies 41 pages.† Its reception was all that could be desired. Reward for the trouble it had cost soon came to hand in the form of addenda from many widely separated correspondents, two of whom, I cannot but recall, examined the scientific serials of their own countries in order to check and supplement the list. An additional list, extending to 22 pages,‡ thus came to be published, and with its publication the preparatory stage was deemed to be over.

The method followed in using the material of the two lists to produce a work, intended to elucidate Determinants by showing the theory in the actual process of growth, is explained in the Introduction (pp. 2–5).

My object, it may be added, has been twofold. First, to provide a work of reference which should contain all that had been written on the subject, and which should be so indexed that any one engaged in research might easily ascertain exactly what had

* Notably that of Schweins. See an article in the *Philosophical Magazine*, vol. xviii. pp. 416–427 (1884), entitled "An Overlooked Discoverer in the Theory of Determinants."

† Vol. xviii. pp. 110–149.

‡ Vol. xxi. pp. 299–320.

been done on any particular topic, how it had been done, and what possible developments it foreshadowed. Secondly, to show clearly to whom every step in advance had been due, doing this in such a way, also, that the reader might see the actual data on which any conclusion was based. Knowing the value of the historical method as a means of *teaching* any branch of science, I cannot but hope that a third result may follow in at least a certain small measure —viz., that some who have never studied Determinants at all may thus readily and in an interesting manner acquire a knowledge of what on all hands is conceded to be a singularly beautiful department of analysis.

My warmest acknowledgments are due to the Members of Council of the Royal Society of Edinburgh, without whose encouragement the work would probably not have been undertaken, and without whose aid it would certainly not have appeared in its present connected form.

T. M.

BEECHCROFT, BOTHWELL,
GLASGOW, *Feb. 15, 1890.*

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HISTORY

OF THE

THEORY OF DETERMINANTS.

PART I.—*Determinants in General.*

In October 1881 I published in the *Quarterly Journal of Mathematics* (xviii. pp. 110–149) a "List of Writings on Determinants," which contained the titles of all the books, pamphlets, memoirs, magazine articles, &c., which were then known to me to exist on the subject of the Theory of Determinants. The list consisted of 489 entries arranged in chronological order, the first date being 1693, and the last 1880. During the three years which have elapsed since it was published, I have been steadily making manuscript additions to it, not merely in the way of continuation for the purpose of keeping it up to date, but also by the intercalation of omitted titles unearthed in the course of my own researches, or brought to my notice by obliging correspondents.

The continuation of the list from 1880 forwards is comparatively an easy matter: it is not by any means easy to render equally complete that portion of the list which pertains to the eighteenth century. In the early history of a scientific subject, before the nomenclature has become fixed, the mere *titles* of writings are insufficient guides: the searcher's work is, consequently, minute and laborious, and he never can be quite sure that his labours are at an end. As far, however, as Determinants are concerned, I am inclined now to think that the writings which are unknown cannot be of much importance, and that the time has come for using the collected material in the production of a detailed history of the subject.

The plan proposed to be followed is not to give one connected history of determinants as a whole, but to give separately the

history of each of the sections into which the subject has been divided, viz., to deal with determinants in general, and thereafter in order with the various special forms. This will not only tend to smoothness in the narrative by doing away with the necessity of frequent harkings back, but it will also be of material importance to investigators who may wish to find out what has already been done in advancing any particular department of the subject. To this end, also, each new result as it appears will be numbered in Roman figures; and if the same result be obtained in a different way, or be generalised, by a subsequent worker, it will be marked among the contributions of the latter with the same Roman figures, followed by an Arabic numeral. Thus the theorem regarding the effect of the transposition of two rows of a determinant will be found under Vandermonde, marked with the number xi., and the information intended thus to be conveyed is that in the order of discovery the said theorem was the *eleventh* noteworthy result obtained: while the mark XI. 2, which occurs under Laplace, is meant to show that the theorem was not then heard of for the first time, but that Laplace contributed something additional to our knowledge of it. In this way any reader who will take the trouble to look up the sequence XI., XI. 2, XI. 3, &c., may be certain, it is hoped, of obtaining the full history of the theorem in question.

The early foreshadowings of a new domain of science, and tentative gropings at a theory of it, are so difficult for the historian to represent without either conveying too much or too little, that the only satisfactory way of dealing with a subject in its earliest stages seems to be to reproduce the exact words of the authors where essential parts of the theory are concerned. This I have resolved to do, although tó some it may have the effect of rendering the account at the commencement somewhat dry and forbidding.

No author, so far as I am aware, has preceded me in the task I have chosen. Sketches of the history have appeared in a number of text-books of the subject, notably in Günther's *Lehrbuch der Determinanten-Theorie für Studirende* (2te Aufl. xii. 209 pp., Erlangen, 1877), which contains a considerable quantity of detail. The *early* history has been very carefully dealt with by F. J. Studnička, in a memoir published in the *Abhandlungen der königl. böhm. Gesellschaft der Wissenschaften*, 6 Folge, viii. 40 pp. (24th

March 1876), and entitled "A. L. Cauchy als formaler Begründer der Determinanten-Theorie. Eine literarisch-historische Studie." There is also an academic thesis (*Teorin för Determinant-Kalkylen*, 121 pp., Helsingfors, 1st March 1876), by E. J. Mellberg, which treats somewhat at length of the early authorities. The existence of these two latter writings has not, however, induced me to curtail to any extent the corresponding part of my work.

LEIBNITZ (1693).

[Leibnizen's mathematische Schriften, herausg. v. C. I. Gerhardt 1 Abth. ii. pp. 229, 238–240, 245, Berlin, 1850.]

In the fourth letter of the published correspondence between Leibnitz and De L'Hospital, the former incidentally mentions that in his algebraical investigations he occasionally uses numbers instead of letters, treating the numbers however as if they were letters. De L'Hospital, in his reply, refers to this, stating that he has some difficulty in believing that numbers can be as convenient or give as general results as letters. Thereupon Leibnitz, in his next letter (28th April 1693), proceeds with an explanation :—

> "Puisque vous dites que vous avés de la peine à croire qu'il soit aussi general et aussi commode de se servir des nombres que des lettres, il faut que je ne me sois pas bien expliqué. On ne sçauroit douter de la generalité en considerant qu'il est permis de se servir de 2, 3, etc., comme d' *a* ou de *b*, pour veu qu'on considere que ce ne sont pas de nombres veritables. Ainsi 2.3 ne signifie point 6 mais autant qu' *ab*. Pour ce qui est de la commodité, il y en a des très grandes, ce qui fait que je m'en sers souvent, sur tout dans les calculs longs et difficiles ou il est aisé de se tromper. Car outre la commodité de l'épreuve par des nombres, et même par l'abjection du novenaire, j' y trouve un tres grand avantage même pour l'avancement de l'Analyse. Comme c'est une ouverture assez extraordinaire, je n'en ay pas encor parlé à d'autres, mais voicy ce que c'est. Lorsqu'on a besoin de beaucoup de lettres, n'est il pas vray que ces lettres n'expriment point les rapports qu'il y a entre les grandeurs qu'elles signifient, au lieu qu'en me servant des nombres je puis exprimer ce rapport. Par exemple soyent

proposées trois equations simples pour deux inconnues à dessein d'oster ces deux inconnues, et cela par un canon general. Je suppose

$$10+11x+12y=0 \qquad (1)$$
$$\text{et} \quad 20+21x+22y=0 \qquad (2)$$
$$\text{et} \quad 30+31x+32y=0 \qquad (3)$$

ou le nombre feint estant de deux characteres, le premier me marque de quelle equation il est, le second me marque à quelle lettre il appartient. Ainsi en calculant on trouve par tout des harmonies qui non seulement nous servent de garans, mais encor nous font entrevoir d'abord des regles ou theoremes. Par exemple ostant premierement y par la premiere et la seconde equation, nous aurons:

$$\begin{array}{l} +10.22+11.22x \\ -12.20-12.21\,.. \end{array} = 0 \qquad (4)^*$$

et par la premiere et troisieme nous aurons:

$$\begin{array}{l} +10.32+11.32x \\ -12.30-12.31\,.. \end{array} = 0 \qquad (5)$$

ou il est aise de connoistre que ces deux equations ne different qu'en ce que le charactere antecedent 2 est changé au charactere antecedent 3. Du reste, dans un même terme d'une même equation les characteres antecedens sont les mêmes, et les characteres posterieurs font une même somme. Il reste maintenant d'oster la lettre x par la quatrieme et cinquieme equation, et pour cet effect nous aurons †

$$\begin{array}{l} 1_0.2_1.3_2 \\ 1_1.2_2.3_0 \\ 1_2.2_0.3_1 \end{array} = \begin{array}{l} 1_0.2_2.3_1 \\ 1_1.2_0.3_2 \\ 1_2.2_1.3_0 \end{array}$$

qui est la derniere equation delivrée des deux inconnues qu'on vouloit oster, et qui porte sa preuve avec soy par les harmonies qui se remarquent par tout, et qu'on auroit bien de la peine à decouvrir en employant des lettres a, b, c, sur tout

* This is written shortly for $\left.\begin{array}{r} +10.22+11.22x=0 \\ -12.20-12.21x=0 \end{array}\right\}$

† The author here slightly changes his notation. What is meant to be is indicated

$10.21.32+11.22.30+12.20.31=10.22.31+11.20.32+12.21.30.$

lors que le nombre des lettres et des equations est grand. Une partie du secret de l'analyse consiste dans la caracteristique, c'est à dire dans l'art de bien employer les notes dont on se sert, et vous voyés, Monsieur, par ce petit echantillon, que Viete et des Cartes n'en ont pas encor connu tous les mysteres. En poursuivant tant soit peu ce calcul on viendra à un *theoreme general* pour quelque nombre de lettres et d'equations simples qu'on puisse prendre. Le voicy comme je l'ay trouvé autres fois :

"*Datis aequationibus quotcunque sufficientibus ad tollendas quantitates, quae simplicem gradum non egrediuntur, pro aequatione prodeunte, primo sumendae sunt omnes combinationes possibiles, quas ingreditur una tantum coefficiens uniuscujusque aequationis : secundo, eae combinationes opposita habent signa, si in eodem aequationis prodeuntis latere ponantur, quae habent tot coefficientes communes, quot sunt unitates in numero quantitatum tollendarum unitate minuto : caeterae habent eadem signa.*

"J'avoue que dans ce cas des degrés simples on auroit peut estre decouvert le même theoreme en ne se servant que de lettres à l'ordinaire, mais non pas si aisement, et ces adresses sont encor bien plus necessaires pour decouvrir des theoremes qui servent à oster les inconnues montées à des degrés plus hauts. Par exemple, ... "

It will be seen that what this amounts to is *the formation of a rule for writing out the resultant of a set of linear equations.* When the problem is presented of eliminating x and y from the equations

$$a+bx+cy=0, \quad d+ex+fy=0, \quad g+hx+ky=0,$$

Leibnitz in effect says that first of all he prefers to write 10 for a, 11 for b, and so on; that, having done this, he can all the more readily take the next step, viz., forming every possible product whose factors are one coefficient from each equation,* the result being

$$10.21.32, \quad 10.22.31, \quad 11.20.32,$$
$$11.22.30, \quad 12.20.31, \quad 12.21.30;$$

and that, then, *one* being the number which is less by one than the

* Of course, this is not exactly what Leibnitz meant to say.

number of unknowns, he makes those terms different in sign which have only *one* factor in common.

The contributions, therefore, which Leibnitz here makes to algebra may be looked upon as three in number :—

(1) A *new notation*, numerical in character and appearance, for individual members of an arranged group of magnitudes; the two numbers which constitute the notation being like the Cartesian co-ordinates of a point in that they denote any one of the said magnitudes by indicating its position in the group, . . . (I.)

(2) A rule for *forming the terms* of the expression which equated to zero is the result of eliminating the unknowns from a set of simple equations, ... (II.)

(3) A rule for *determining the signs* of the terms in the said result. . . . . (III.)

The last of these is manifestly the least satisfactory. In the first place, part of it is awkwardly stated. Making those terms different in sign *which have only as many factors alike as is indicated by the number which is less by one than the number of unknown quantities* is exactly the same as making those terms different in sign *which have only two factors different.* Secondly, in form it is very unpractical. The only methodical way of putting it in use is to select a term and make it positive; then seek out a second term, having all its factors except two the same as those of the first term, and make this second term negative; then seek out a third term, having all its factors except two the same as those of the second term, and make this third term positive; and so on.

Although there is evidence that Leibnitz continued, in his analytical work, to use his new notation for the coefficients of an equation (see Letters xi., xii., xiii. of the said correspondence), and that he thought highly of it (see Letter viii. "chez moi c'est une des meilleures ouvertures en Analyse"), it does not appear that by using it in connection with sets of linear equations, or by any other means, he went further on the way towards the subject with which we are concerned. Moreover, it must be remembered that the little he did effect had no influence on succeeding workers. So far as is known, the passage above quoted from his correspondence with De L'Hospital was not published until 1850. Even for some little

time after the date of Gerhardt's publication it escaped observation, Lejeune Dirichlet being the first to note its historical importance. It is true that during his own lifetime, Leibnitz's *use of numbers in place of letters* was made known to the world in the *Acta Eruditorum* of Leipzig for the year 1700 (*Responsio ad Dn. Nic. Fatii Duillerii imputationes*, pp. 189–208); but the particular application of the new symbols which brings them into connection with determinants was not there given.

CRAMER (1750).

[Introduction a l'Analyse des Lignes Courbes algébriques, par Gabriel Cramer, pp. 59, 60, 656–659. Génève, 1750.]

The third chapter of Cramer's famous treatise deals with the different *orders* (degrees) of curves, and one of the earliest theorems of the chapter is the well-known one that the equation of a curve of the nth degree is determinable when $\frac{1}{2}n(n+3)$ points of the curve are known. In illustration of this theorem he deals (p. 59) with the case of finding the equation of the curve of the *second* degree which passes through *five* given points. The equation is taken in the form

$$A + By + Cx + Dyy + Exy + xx = 0 ;$$

the five equations for the determination of A, B, C, D, E are written down; and it is pointed out that all that is necessary is the solution of the set of five equations, and the substitution of the values of A, B, C, D, E thus found. "Le calcul véritablement en seroit assez long," he says; but in a footnote there is the remark that it is to algebra we must look for the means of shortening the process, and we are directed to the appendix for a convenient general rule which he had discovered for obtaining the solution of a set of equations of this kind. The following is the essential part of the passage in which the rule occurs:—

"Soient plusieurs inconnues z, y, x, v, &c., et autant d'équations

$$\begin{aligned}
A^1 &= Z^1z + Y^1y + X^1x + V^1v + \&c. \\
A^2 &= Z^2z + Y^2y + X^2x + V^2v + \&c. \\
A^3 &= Z^3z + Y^3y + X^3x + V^3v + \&c. \\
A^4 &= Z^4z + Y^4y + X^4x + V^4v + \&c. \\
&\quad \&c.
\end{aligned}$$

où les lettres A^1, A^2, A^3, A^4, &c., ne marquent, pas comme à l'ordinaire, les puissances d' A, mais le prémier membre, supposé connu, de la prémiére, seconde, troisiéme, quatriéme, &c. équation."

[Here the solutions of the cases of 1, 2, and 3 unknowns are given, and he then proceeds.]

"L'examen de ces Formules fournit cette Règle générale. Le nombre des équations et des inconnues étant n, on trouvera la valeur de chaque inconnue en formant n fractions dont le dénominateur commun à autant de termes qu'il y a de divers arrangements de n choses différentes. Chaque terme est composé des lettres ZYXV, &c., toujours écrites dans le même ordre, mais auxquelles on distribue, comme exposants, les n prémiers chiffres rangés en toutes les maniéres possibles. Ainsi, lorsqu'on a trois inconnues, le dénominateur a [$1 \times 2 \times 3 =$] 6 termes, composés des trois lettres ZYX, qui recoivent successivement les exposants 123, 132, 213, 231, 312, 321. On donne à ces termes les signes + ou −, selon la Règle suivante. Quand un exposant est suivi dans le même terme, médiatement ou immédiatement, d'un exposant plus petit que lui, j'appellerai cela un *dérangement*. Qu'on compte, pour chaque terme, le nombre des dérangements : s'il est pair ou nul, le terme aura le signe +; s'il est impair, le terme aura le signe −. Par ex. dans le terme $Z^1Y^2V^3$ il n'y a aucun dérangement; ce terme aura donc le signe +. Le terme $Z^3Y^1X^2$ a aussi le signe +, parce qu'il a deux dérangements, 3 avant 1 et 3 avant 2. Mais le terme $Z^3Y^2X^1$, qui a trois dérangements, 3 avant 2, 3 avant 1, et 2 avant 1, aura le signe −.

"Le dénominateur commun étant ainsi formé, on aura la valeur de z en donnant à ce dénominateur le numérateur qui se forme en changeant, dans tous ces termes, Z en A. Et la valeur d' y est la fraction qui a le même dénominateur et pour numérateur la quantité qui résulte quand on change Y en A, dans tous les termes du dénominateur. Et on trouve d'une maniére semblable la valeur des autres inconnues."

It is evident at once that the new results here given are—

(1) A rule for *forming the terms* of the common denominator of

the fractions which express the values of the unknowns in a set of linear equations, ... (IV.)

(2) A rule for *determining the sign* of any individual term in the said common denominator (and, included in the rule, the notion of a "dérangement"), ... (III. 2)

(3) A rule for *obtaining the numerators* from the expression for common denominator, ... (V.)

The problem which Cramer set himself at this point in his book was exactly that which Leibnitz had solved, viz., the elimination of n quantities from a set of $n+1$ linear equations. The solution which Cramer obtained, and which, be it remarked, was the solution best adapted for his purpose, was quite distinct in character from that of Leibnitz. Leibnitz gave a rule for writing out the final result of the elimination; what Cramer gives is a rule for writing out the values of the n unknowns as determined from n of the $n+1$ equations, after which we have got to substitute these values in the remaining $(n+1)$th equation. The notable point in regard to the two solutions is, that Cramer's rule for writing the *common denominator* of the values of the n unknowns (an expression of the nth degree in the coefficients) is exactly Leibnitz's rule for writing the *final result*, which is an expression of the $(n+1)$th degree. Had either discoverer been aware that the same rule sufficed for obtaining both of these expressions, he could not have failed, one would think, to note the *recurrent* law of formation of them. The result of eliminating w, x, y, z from the equations,

$$a_rw + b_rx + c_ry + d_rz = e_r \quad (r = 1,\ 2,\ 3,\ 4,\ 5)$$

is, according to Leibnitz, if we embody his rule in a later symbolism,

$$|\, a_1b_2c_3d_4e_5 \,| = 0\,;$$

whereas, according to Cramer, it is—

$$a_1\frac{|e_2b_3c_4d_5|}{|a_2b_3c_4d_5|} + b_1\frac{|a_2e_3c_4d_5|}{|a_2b_3c_4d_5|} + c_1\frac{|a_2b_3e_4d_5|}{|a_2b_3c_4d_5|} + d_1\frac{|a_2b_3c_4e_5|}{|a_2b_3c_4d_5|} = e_1,$$

and from the collocation of these the one natural step is to the identity

$$-|a_1b_2c_3d_4e_5| = a_1|e_2b_3c_4d_5| + b_1|a_2e_3c_4d_5| + \ldots\ldots - e_1|a_2b_3c_4d_5|\,.$$

The fate of Cramer's rule was very different from that of Leibnitz.

It was soon taken up, and after a time found its way into the schools, where it continued for many years to be taught as the nutshell form of the theory of the solution of simultaneous linear equations. Indeed Gergonne is reported* to have said, "Cette methode était tellement en faveur, que les examens aux écoles des services publics ne roulaient, pour ainsi dire, que sur elle ; on était admis ou rejeté suivant qu'on la possedait bien ou mal."

Finally, the exact difference between Cramer's notation for the coefficients of the unknowns and the notation of Leibnitz should be noted, and in connection therewith the fact that when dealing with the subject of elimination between two equations of the *m*th and *n*th degrees in x Cramer uses a notation closely resembling that which Leibnitz employed, viz. $[1^2]$ $[1^3]$, &c.

BÉZOUT (1764).

[Recherches sur le degré des équations résultantes de l'évanouissement des inconnues, et sur les moyens qu'il convient d'employer pour trouver ces équations.—*Hist. de l'Acad. Roy. des Sciences*, Ann. 1764 (pp. 288–338), pp. 291–295.]

The object of Bézout's memoir is sufficiently apparent from the title; we may therefore at once give those portions of it which directly concern our subject. On p. 291 is the commencement of the following passage:—

"M. Cramer a donné une règle générale pour les exprimer toutes débarrassées de ce facteur: j'aurois pu m'en tenir à cette règle; mais l'usage m'a fait connoître que quoiqu'elle soit assez simple, quant aux lettres, elle ne l'est pas de même à l'égard des signes lorsqu'on a au-delà d'un certain nombre d'inconnues à calculer ; . . . .

Lemme I.

"Si l'on a un nombre n d'équations du premier degré qui renferment chacune un pareil nombre d'inconnues, sans aucun terme absolument connu, on trouvera par la règle suivante la relation que doivent avoir les coëfficiens de ces inconnues pour que toutes ces équations aient lieu.

* By Studnička.

"Soient a, b, c, d, &c., les coëfficiens de ces inconnues dans la première équation.

a', b', c', d', &c., les coëfficiens des mêmes inconnues dans la seconde équation.

a'', b'', c'', d'', &c., ceux de la troisième & ainsi de suite.

"Formez les deux permutations ab & ba & écrivez $ab - ba$; avec ces deux permutations & la lettre c formez toutes les permutations possibles, en observant de changez de signe toutes les fois que c changera de place dans ab & la même chose à l'égard de ba; vous aurez

$$abc - acb + cab - bac + bca - cba.$$

Avec ces six permutations & la lettre d, formez toutes les permutations possibles, en observant de changer de signe à chaque fois que d changera de place dans un même terme; vous aurez

$$\begin{gathered} abcd - abdc + adbc - dabc - acbd + acdb - adcb + dacb \\ + cabd - cadb + cdab - dcab - bacd + badc - bdac + dbac \\ + bcad - bcda + bdca - dbca - cbad + cbda - cdba + dcba \end{gathered}$$

& ainsi de suite jusqu'à ce que vous ayez épuisé tous les coëfficiens de la première equation.

" Alors conservez les lettres qui occupent la première place; donnez à celles qui occupent la seconde, la même marque qu'elles ont dans la seconde équation; à celles qui occupent la troisième, la même marque qu'elles ont dans la troisième équation, & ainsi de suite; égalez enfin le tout à zéro et vous aurez l'équation de condition cherchée.

" Ainsi si vous avez deux équations et deux inconnues comme

$$\begin{aligned} ax + by &= 0 \\ a'x + b'y &= 0 \end{aligned}$$

l'équation de condition sera $ab' - ba' = 0$ ou $ab' - a'b = 0 \ldots$."

In the same way the next two cases are given; then—

" . . . . mais comme ces équations de condition doivent servir de formules pour l'élimination dans les équations de différens degrés, il convient de leur donner une forme qui

rende les substitutions le moins pénibles qu'il se pourra ; pour cet effet, je les mets sous cette forme :

$$ab' - a'b = 0$$

$$(ab' - a'b)c'' + (a''b - ab'')c' + (a'b'' - a''b')c = 0$$

$$\begin{aligned} &[(ab' - a'b)c'' + (a''b - ab'')c' + (a'b'' - a''b')c\,]d''' \\ +&[(a'b - ab')c''' + (ab''' - a'''b)c' + (a'''b' - a'b''')c\,]d'' \\ +&[(a'''b - ab''')c'' + (ab'' - a''b)c''' + (a''b''' - a'''b'')c]d' \\ +&[(a'b''' - a'''b')c'' + (a'''b'' - a''b''')c' + (a''b' - a'b'')c''']d = 0. \end{aligned}$$

Cette nouvelle forme a deux avantages : le premier, de rendre les substitutions à venir, plus commodes ; le deuxième, c'est d'offrir une règle encore plus simple pour la formation de ces formules.

"En effet, il est facile de remarquer 1°, que le premier terme de l'une quelconque de ces équations, est formé du premier membre de l'équation précédente, multiplié par la première des lettres qu'elle ne renferme point, cette lettre étant affectée de la marque qui suit immédiatement la plus haute de celles qui entrent dans ce même membre.

"2°. Le deuxième terme se forme du premier, en changeant dans celui-ci la plus haute marque en celle qui est immédiatement au-dessous & réciproquement, and de plus en changeant les signes.

"3°. Le troisième, se forme du premier, en changeant dans celui-ci la plus haute marque en celle de deux numéros au-dessous & réciproquement, & de plus en changeant les signes.

"4°. Le quatrième, se forme du premier, en changeant dans celui-ci la plus haute marque en celle de trois numéros au-dessous & réciproquement, & changeant les signes, & toujours de même pour les suivans.

"Par exemple, ...

"D'après ces observations, il sera facile de voir que l'équation de condition pour cinq inconnues et cinq équations, sera

. . . ."

The latter part of this we are drawn to at once, as it enunciates quite clearly the Recurrent Law of Formation to which attention has above been directed as a natural deduction from the work of Leibnitz and Cramer.

The notable point in regard to the earlier portion is, that Bézout throws his rule of term-formation and his rule of signs into one. In the case of finding the resultant of

$$a_r x + b_r y + c_r z = 0 \qquad (r = 1, 2, 3)$$

his process consists of four steps, viz. :—

(1) a.

(2) $a\,b \;\vdots - b\,a$

(3) $a\,b\,c\; - a\,c\,b\; + c\,a\,b \;\vdots - b\,a\,c\; + b\,c\,a\; - c\,b\,a$.

(4) $a_1b_2c_3 - a_1c_2b_3 + c_1a_2b_3 - b_1a_2c_3 + b_1c_2a_3 - c_1b_2a_3$.

The first term of (2) is got from (1) by affixing b, and the second is got from the first by advancing the b one place and changing the sign. The first term of (3) is got from the first term of (2) by affixing c; the second term is got from the first by advancing c a place and changing the sign, and the third is got from the second by advancing c a place and changing the sign; the last three are got from the second term of (2) in the same way as the first three are got from the first term of (2).

It will thus be seen that while Leibnitz and Cramer direct us to find the permutations in any way whatever, and thereafter to fix the sign of each in accordance with a rule, Bézout requires the permutations to be found by a particular process, and attention given to the question of sign throughout all this process, so that when the terms have been found their signs have likewise been determined.

Bézout's contributions to the subject thus are —

(1) A combined rule of term-formation and rule of signs, ... } (II. 2) + (III. 3)

(2) The recurrent law of formation of the new functions, (VI.)

VANDERMONDE (1771).

[Mémoire sur l'élimination. *Hist. de l'Acad. Roy. des Sciences.* Ann. 1772, 2ᵉ partie (pp. 516–532).]

This important memoir of Vandermonde and that of Laplace, which is dealt with immediately afterwards, both appear in the

History of the French Academy of Sciences for 1772, Laplace's memoir occupying pp. 267–376, and Vandermonde's pp. 516–532. There is, however, a footnote to the latter, which states that it was read for the first time to the Academy on 12th January 1771.

The part of it which concerns us is the first article, which treats of elimination in the case of equations of the first degree. Vandermonde here writes :—

"Je suppose que l'on représente par $\overset{1}{1},\ \overset{2}{1},\ \overset{3}{1},$ &c., $\overset{1}{2},\ \overset{2}{2},\ \overset{3}{2},$ &c., $\overset{1}{3},\ \overset{2}{3},\ \overset{3}{3},$ &c., &c., autant de différentes quantités générales, dont l'une quelconque soit $\overset{\alpha}{a},$ une autre quelconque soit $\overset{\beta}{b},$ &c., & que le produit des deux soit désigné à l'ordinaire par $\begin{array}{l}\alpha\ \beta\\ a\,.\,b.\end{array}$

"Des deux nombres ordinaux α & a, le premier, par exemple, désignera de quelle équation est pris le coëfficient $\overset{\alpha}{a},$ & le second désignera le rang que tient ce coëfficient dans l'équation, comme on le verra ci-après.

"Je suppose encore le système suivant d'abréviations, & que l'on fasse

$$\begin{array}{c|c}\alpha&\beta\\\hline a&b\end{array} = \begin{array}{c}\alpha\ \beta\\ a.b\end{array} - \begin{array}{c}\alpha\ \beta\\ b.a\end{array},$$

$$\begin{array}{c|c|c}\alpha&\beta&\gamma\\\hline a&b&c\end{array} = \overset{\alpha}{a}.\begin{array}{c|c}\beta&\gamma\\\hline b&c\end{array} + \overset{\alpha}{b}.\begin{array}{c|c}\beta&\gamma\\\hline c&a\end{array} + \overset{\alpha}{c}.\begin{array}{c|c}\beta&\gamma\\\hline a&b\end{array},$$

$$\begin{array}{c|c|c|c}\alpha&\beta&\gamma&\delta\\\hline a&b&c&d\end{array} = \overset{\alpha}{a}.\begin{array}{c|c|c}\beta&\gamma&\delta\\\hline b&c&d\end{array} - \overset{\alpha}{b}.\begin{array}{c|c|c}\beta&\gamma&\delta\\\hline c&d&a\end{array} + \overset{\alpha}{c}.\begin{array}{c|c|c}\beta&\gamma&\delta\\\hline d&a&b\end{array} - \overset{\alpha}{d}.\begin{array}{c|c|c}\beta&\gamma&\delta\\\hline a&b&c\end{array}$$

$$\begin{array}{c|c|c|c|c|}\alpha&\beta&\gamma&\delta&\epsilon\\\hline a&b&c&d&e\end{array} = \overset{\alpha}{a}.\begin{array}{c|c|c|c}\beta&\gamma&\delta&\epsilon\\\hline b&c&d&e\end{array} + \ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

"Le symbole $\begin{array}{c|c|c}&&\\\hline &&\end{array}$ sert ici de caractéristique. Les seules choses à observer sont l'ordre des signes, et la loi des permutations entre les lettres $a, b, c, d,$ &c., qui me paroissent suffisamment indiquées ci-dessus.

"Au lieu de transposer les lettres $a, b, c, d,$ &c., on pouvoit

les laisser dans l'ordre alphabétique, & transposer au contraire les lettres α, β, γ, δ, &c., les résultats auroient été parfaitement les mêmes; ce qui a lieu aussi par rapport aux conclusions suivantes.

"Premièrement, il est clair que $\dfrac{\alpha \mid \beta}{a \mid b}$ représente deux termes différens, l'un positif, & l'autre négatif, résultans d'autant de permutations possibles de a & b; que $\dfrac{\alpha \mid \beta \mid \gamma}{a \mid b \mid c}$ en représente six, trois positifs & trois négatifs, résultans d'autant de permutations possibles de a, b, & c; que $\dfrac{\alpha \mid \beta \mid \gamma \mid \delta}{a \mid b \mid c \mid d}$ ....

"Mais de plus, la formation de ces quantités est telle que l'unique changement que puisse résulter d'une permutation, quelle qu'elle soit, faite entre les lettres du même alphabet, dans l'une de ces abréviations, sera un changement dans le signe de la première valeur.

"La démonstration de cette vérité & la recherche du signe résultant d'une permutation déterminée, dépendent généralement de deux propositions qui peuvent être énoncées ainsi qu'il suit, en se servant de nombres pour indiquer le rang des lettres.

"La première est que

$$\frac{1 \mid 2 \mid 3 \mid \ldots \mid m \mid m+1 \mid \ldots n}{1 \mid 2 \mid 3 \mid \ldots\ m \mid m+1 \mid \ldots n}$$

$$= \pm \frac{1 \mid 2 \mid 3 \mid \ldots \mid n-m+1 \mid n-m+2 \mid n-m+3 \mid \ldots \mid n}{m \mid m+1 \mid m+2 \mid \ldots \mid n \mid 1 \mid 2 \mid \ldots \mid m-1}$$

le signe − n'ayant lieu que dans le cas où n & m sont l'un & l'autre des nombres pairs.

"La seconde est que

$$\frac{1 \mid 2 \mid 3 \mid \ldots \mid m \mid m+1 \mid \ldots \mid n}{1 \mid 2 \mid 3 \mid \ldots \mid m \mid m+1 \mid \ldots \mid n}$$

$$= -\frac{1 \mid 2 \mid 3 \mid \ldots \mid m-1 \mid m \mid m+1 \mid m+2 \mid \ldots \mid n}{1 \mid 2 \mid 3 \mid \ldots \mid m-1 \mid m+1 \mid m \mid m+2 \mid \ldots \mid n}.$$

"Il sera facile de voir que, la première équation supposée,

celle-ci n'a besoin d'être prouvée que pour un seul cas, comme, par exemple, celui de $m=n-1$, c'est-à-dire, celui où les deux lettres transposées sont les deux dernières.

" Au lieu de démontrer généralement ces deux équations, ce qui exigeroit un calcul embarrassant plutôt que difficile, je me contenterai de développer les exemples les plus simples : cela suffira pour saisir l'esprit de la démonstration.

. . . ."

(2½ pages are occupied with verifications for the case of

$$\frac{\alpha|\beta}{a|b}, \text{ of } \frac{\alpha|\beta|\gamma}{a|b|c}, \text{ and of } \frac{\alpha|\beta|\gamma|\delta}{a|b|c|d}.)$$

" On verra qu'en général la démonstration de notre seconde équation pour le cas $n=a$, dépend de cette même équation pour le cas $n=a-1$, quel que soit a : d'où il suit que puisque $\frac{1|2}{1|2}=-\frac{1|2}{2|1}$, elle est généralement vraie.

. . . .

" De ce que nous avons dit jusqu'ici *il suit que*

$$\frac{\alpha|\beta|\gamma|\delta|\ldots\ldots}{a|b|c|d|\ldots\ldots}=0,$$

si deux lettres quelconques du même alphabet sont égales entr'elles ; car quelque part que soient les deux lettres égales, on peut les transposer aux deux dernières places de leur rang, ce qui ne fera au plus que changer le signe de la valeur ; alors, de leur permutation particulière, il ne peut, d'une part, résulter aucun changement, puisqu'elles sont égales ; d'autre part, selon notre seconde équation ci-dessus, il doit en résulter un changement de signe ; cette contradiction ne peut être levée qu'en supposant la valeur zéro. . . . . . .

" Tout cela posé ; puisque l'on a identiquement,

$$\frac{1|1|2}{1|2|3}=\frac{1}{1}.\frac{1|2}{2|3}+\frac{1}{2}.\frac{1|2}{3|1}+\frac{1}{3}.\frac{1|2}{1|2}=0,$$

$$\frac{2|1|2}{1|2|3}=\frac{2}{1}.\frac{1|2}{2|3}+\frac{2}{2}.\frac{1|2}{3|1}+\frac{2}{3}.\frac{1|2}{1|2}=0,$$

si l'on propose de trouver les valeurs de $\xi 1$ et de $\xi 2$ qui satisfont aux deux équations

$$\overset{1}{1}\,.\,\xi 1 + \overset{2}{2}\,.\,\xi 2 + \overset{1}{3} = 0$$

$$\overset{2}{1}\,.\,\xi 1 + \overset{2}{2}\,.\,\xi 2 + \overset{2}{3} = 0,$$

on pourra comparer, & l'on aura

$$\xi 1 = \frac{\dfrac{1\,|\,2}{2\,|\,3}}{\dfrac{1\,|\,2}{1\,\;2}}, \quad \xi 2 = \frac{\dfrac{1\,|\,2}{3\,|\,1}}{\dfrac{1\,|\,2}{1\,|\,2}}.$$

. . . ."

(Three equations with three unknowns similarly dealt with.)

" Il est clair que ces valeurs n'ont point de facteurs inutiles: mais pour les rendre aussi commodes qu'il est possible dans les applications, and particulièrement dans celles où l'on veut faire usage des logarithmes, il sera bon d'y employer le plus qu'il se pourra, la multiplication des facteurs complexes. J'observe donc 1° que si l'on substitue dans le développement de $\dfrac{\alpha\,|\,\beta\,|\,\gamma\,|\,\delta}{a\,|\,b\,|\,c\,|\,d}$, les valeurs des $\dfrac{\alpha\,|\,\beta\,|\,\gamma}{a\,|\,b\,|\,c}$ en $\dfrac{\alpha\,|\,\beta}{a\,|\,b}$, on aura, en réduisant & ordonnant, d'après les observations ci-dessus,

$$\frac{\alpha\,|\,\beta\,|\,\gamma\,|\,\delta}{a\,|\,b\,|\,c\,|\,d} = \left\{\begin{array}{l} \dfrac{\alpha\,|\,\beta}{a\,|\,b}\cdot\dfrac{\gamma\,|\,\delta}{c\,|\,d} - \dfrac{\alpha\,|\,\beta}{a\,|\,c}\cdot\dfrac{\gamma\,|\,\beta}{b\,|\,d} + \dfrac{\alpha\,|\,\beta}{a\,|\,d}\cdot\dfrac{\gamma\,|\,\delta}{b\,|\,c} \\ + \dfrac{\alpha\,|\,\beta}{b\,|\,c}\cdot\dfrac{\gamma\,|\,\delta}{a\,|\,d} - \dfrac{\alpha\,|\,\beta}{b\,|\,d}\cdot\dfrac{\gamma\,|\,\delta}{a\,|\,c} \\ + \dfrac{\alpha\,|\,\beta}{c\,|\,d}\cdot\dfrac{\gamma\,|\,\delta}{a\,|\,b} \end{array}\right.$$

si de même on substitue dans le développement des $\dfrac{\alpha\,|\,\beta\,|\,\gamma\,|\,\delta\,|\,\epsilon\,|\,\zeta}{a\,|\,b\,|\,c\,|\,d\,|\,e\,|\,f}$, les valeurs des $\dfrac{\alpha\,|\,\beta\,|\,\gamma\,|\,\delta\,|\,\epsilon}{a\,|\,b\,|\,c\,|\,d\,|\,e}$ en $\dfrac{\alpha\,|\,\beta\,|\,\gamma\,|\,\delta}{a\,|\,b\,|\,c\,|\,d}$, on aura, en réduisant & ordonnant, d'après les observations ci-dessus,

$$\frac{\alpha|\beta|\gamma|\delta|\epsilon|\zeta}{a|b|c|d|e|f}=\left\{\begin{aligned}
&\frac{\alpha|\beta}{a|b}\cdot\frac{\gamma|\delta|\epsilon|\zeta}{c|d|e|f}-\frac{\alpha|\beta}{a|c}\cdot\frac{\gamma|\delta|\epsilon|\zeta}{b|d|e|f}+\frac{\ |\ }{a|d}\cdot\frac{|\ |\ |}{b|c|e|f}\\
&+\frac{\alpha|\beta}{b|c}\cdot\frac{\gamma|\delta|\epsilon|\zeta}{a|d|e|f}-\frac{\ |\ }{b|d}\cdot\frac{|\ |\ |}{a|c|e|f}+\frac{\ |\ }{b|e}\cdot\frac{|\ |\ |}{a|c|d|f}\\
&+\frac{\ |\ }{c|d}\cdot\frac{|\ |\ |}{a|b|e|f}-\frac{\ |\ }{c|e}\cdot\frac{|\ |\ |}{a|b|d|f}+\frac{\ |\ }{c|f}\cdot\frac{|\ |\ |}{a|b|d|e}\\
&+\frac{\ |\ }{d|e}\cdot\frac{|\ |\ |}{a|b|c|f}-\frac{\ |\ }{d|f}\cdot\frac{|\ |\ |}{a|b|c|e}\\
&+\frac{\ |\ }{e|f}\cdot\frac{|\ |\ |}{a|b|c|d}\\
&\qquad\qquad\qquad-\frac{\ |\ }{a|e}\cdot\frac{|\ |\ |}{b|c|d|f}+\frac{\ |\ }{a|f}\cdot\frac{|\ |\ |}{b|c|d|e}\\
&\qquad\qquad\qquad-\frac{\ |\ }{b|f}\cdot\frac{|\ |\ |}{a|c|d|e}
\end{aligned}\right.$$

"La loi des permutations & des signes est assez manifeste dans ces exemples, pour qu'on en puisse conclure des développemens pareils pour les cas de huit & dix lettres, &c., du même alphabet; alors, en employant les premiers développemens pour les cas d'un nombre impair de ces lettres, on aura les formules d'élimination du premier degré, sous la forme la plus concise qu'il soit possible.

"Si l'on veut exprimer ces formules, généralement pour un nombre n d'équations

$$\overset{1}{1}.\xi 1+\overset{1}{2}.\xi 2+\overset{1}{3}.\xi 3+\ldots+\overset{1}{m}.\xi m+\ldots+\overset{1}{n}.\xi n+(n\overset{1}{+}1)=0$$

$$\overset{2}{1}.\xi 1+\overset{2}{2}.\xi 2+\overset{2}{3}.\xi 3+\ldots+\overset{2}{m}.\xi m+\ldots+\overset{2}{n}.\xi n+(n\overset{2}{+}1)=0$$

&c.

la valeur de l'inconnue quelconque ξm, sera renfermée dans l'équation suivante, à une seule inconnue

$$\frac{1\,|\,2\,|\,3\,|\ldots|\,n}{1\,|\,2\,|\,3\,|\ldots|\,n}.\,\xi m$$

$$\pm\frac{1\ |\ 2\ |\ 3\ |\ldots\ldots\ldots|n-m|n-m+1|n-m+2|n-m+3|\ldots|\ n}{m+1|m+2|m+3|\ldots\ldots\ldots|\ n\ |\ n+1\ |\ 1\ |\ 2\ |\ m-1}=0$$

le signe + ayant lieu seulement dans le cas où m & n sont impairs l'un & l'autre."

Taking this up in order, we observe first that Vandermonde' pro-

poses for coefficients a positional notation essentially the same as that of Leibnitz, writing $\frac{1}{2}$ where Leibnitz wrote 12 or 1_2.

Then he defines a certain class of functions by means of their recurrent law of formation—a law and class of functions at once seen to be identical with those of Bézout. A special symbolism is used for the first time to denote the functions; thus, the expression

$$1_0.2_1.3_2 + 1_1.2_2.3_0 + 1_2.2_0.3_1 - 1_0.2_2.3_1 - 1_1.2_0.3_2 - 1_2.2_1.3_0,$$

which occurs in Leibnitz's letter, Vandermonde would have denoted by

$$\frac{1\,|\,2\,|\,3}{1\,|\,2\,|\,3},$$

and the result of eliminating x, y, z, w from the set of equations

$$1_r x + 2_r y + 3_r y + 4_r w = 0 \quad (r = 1, 2, 3, 4)$$

by

$$\frac{1\,|\,2\,|\,3\,|\,4}{1\,|\,2\,|\,3\,|\,4}.$$

It is next pointed out that permutation of the under row of indices produces the same result as permutation of the upper row, that the number of terms is the same as the number of permutations of either row of indices, and that half of the terms are positive and half negative.

The part which follows this is a little curious. The proposition is brought forward that if in the symbolism for one of the functions a transposition of indices takes place in either row, the same function is still denoted, the only change thereby possible being a change of sign. The demonstration is affirmed to be dependent on two theorems, neither of which is proved, as the proofs are said to be troublesome to set forth. Now it will be seen that the second of these theorems is to the effect that the transposition of any two consecutive indices causes a change of sign, and that consequently this alone is sufficient for the required demonstration. The first of the auxiliary theorems, in fact, is an immediate deduction from the second, the particular permutation which it concerns being produced by $(n-m+1)(m-1)$ transpositions of pairs of consecutive indices.

Passing over the illustrations of these propositions, we come next to the theorem that if any two indices of either row be equal the

function vanishes identically, and we note particularly that the basis of the proof is that the interchange of the two indices in question changes the sign of the function, and yet leaves the function unaltered.

Upon this theorem the solution of a set of simultaneous linear equations is then with much neatness made to depend. In more modern notation Vandermonde's process is as follows:—It is known that

$$a_1 \mid b_1c_2 \mid + b_1 \mid c_1a_2 \mid + c_1 \mid a_1b_2 \mid = \mid a_1b_1c_2 \mid = 0,$$

and

$$a_2 \mid b_1c_2 \mid + b_2 \mid c_1a_2 \mid + c_2 \mid a_1b_2 \mid = \mid a_2b_1c_2 \mid = 0,$$

$$\left.\begin{aligned} \therefore\quad & a_1\frac{\mid b_1c_2 \mid}{\mid a_1b_2 \mid} + b_1\frac{\mid c_1a_2 \mid}{\mid a_1b_2 \mid} + c_1 = 0 \\ \text{and}\quad & a_2\frac{\mid b_1c_2 \mid}{\mid a_1b_2 \mid} + b_2\frac{\mid c_1a_2 \mid}{\mid a_1b_2 \mid} + c_2 = 0 \end{aligned}\right\}$$

hence, if the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{aligned}\right\}$$

be given us, we know that

$$x = \frac{\mid b_1c_2 \mid}{\mid a_1b_2 \mid},\quad y = \frac{\mid c_1a_2 \mid}{\mid a_1b_2 \mid}$$

is a solution.

This result, moreover, is generalised; the solution of

$$r_1x_1 + r_2x_2 + \ldots\ldots + r_nx_n + r_{n+1} = 0 \qquad (r = 1, 2, \ldots, n)$$

being fully and accurately expressed in symbols, although the numerators of the values of $x_1, x_2, \ldots, x_n$ are not in so simple a form as Cramer's rule for obtaining the numerator from the denominator might have suggested.

Lastly, and almost incidentally, Vandermonde makes known a case of the widely general theorem now-a-days described as the theorem for expressing a determinant as an aggregate of products of complementary minors. His case is that in which the given determinant is of the order $2m$, and one factor of each of the products is of order 2.

Summing up, therefore, we must put the statement of our indebtedness to Vandermonde as follows:—

(1) A simple and appropriate notation for the new functions, *e.g.*,

$$\frac{1\mid 2\mid 3}{1\mid 2\mid 3}, \quad \ldots\ldots \quad \text{(VII.)}$$

(2) A new mode of defining the functions, viz., using Bézout's recurring law of formation, ... (VIII.)

(3) The remark that the ordinary algebraical expression of any of the functions is obtainable by permutation of *either* series of indices, ... (IX.)

(4) The remark that the positive and negative terms are equal in number, ... (X.)

(5) The theorem regarding the effect of interchanging two consecutive indices, ... (XI.)

(6) The theorem (with proof) regarding the effect of equality of two indices belonging to the same series, ... (XII.)

(7) A reasoned-out solution of a set of n simultaneous linear equations, by means of the new functions as above defined, . (XIII.)

(8) Expression of any of the new functions of order $2m$ as an aggregate of products of like functions of orders 2 and $2m-2$, ... (XIV.)

In addition to this, we must view Vandermonde's work as a whole, and note that he is the first to give a connected exposition of the theory, defining the functions apart from their connections with other matter, assigning them a notation, and thereafter logically developing their properties. After Vandermonde there could be no absolute necessity for a renovation or reconstruction on a new basis: his successors had only to extend what he had done, and, it might be, to perfect certain points of detail. Of the mathematicians whose work has thus far been passed in review, the only one fit to be viewed as the founder of the theory of determinants is Vandermonde.

LAPLACE (1772).

[Recherches sur le calcul intégral et sur le système du monde. *Hist. de l'Acad. Roy. des Sciences, Ann.* 1772, 2^e *partie* (pp. 267–376) pp. 294–304].

In the course of his work Laplace arrives at a set of linear equations from which n quantities have to be eliminated. This he says

can be accomplished by means of rules which mathematicians have given :—

> "Mais comme elles ne me paroissent avoir été jusqu'ici démontrées que par induction, et que d'ailleurs elles sont impracticables, pour peu que le nombre des équations soit considérable; je vais reprendre de nouveau cette matière, et donner quelques procédés plus simples que ceux qui sont déjà connus, pour éliminer entre un nombre quelconque d'équations du premier degré."

Taking n homogeneous linear equations with the coefficients

$$\begin{array}{llll} {}^1a, & {}^1b, & {}^1c, & \ldots \\ {}^2a, & {}^2b, & {}^2c, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

he first gives Cramer's rule for writing out what he, Laplace, calls the *Resultant,* using in the course of the rule the term *variation* instead of Cramer's term "*dérangement.*" Then he gives the "perhaps simpler" rule of Bézout, and shows that of necessity it will lead to the same result as Cramer's.

The theorem in regard to the effect of transposing two letters is next enunciated, and the blank left by Vandermonde is filled, for a proof of the theorem is given. The exact words of the enunciation and proof are—

> "Si au lieu de combiner d'abord la lettre a avec la lettre b, ensuite ces deux-ci avec la lettre c, et ainsi de suite; c'est-à-dire, si au lieu de combiner les lettres a, b, c, d, e, &c., dans l'ordre a, b, c, d, e, &c., on les eût combinées dans l'ordre a, c, b, d, e, &c., ou a, d, b, c, e, &c., ou a, e, b, c, d, &c., ou &c., je dis qu'on auroit toujours eu la même quantité à la différence des signes près.
>
> "Pour démontrer ce Théorème nommons en genéral, *resultante,* la quantité qui résulte de l'une quelconque de ces combinaisons, en sorte que la *première résultante* soit celle qui vient de la combinaison suivant l'ordre a, b, c, d, e, &c., que la *seconde résultante* soit celle qui vient de la combinaison suivant l'ordre a, c, b, d, e, &c., que la *troisième résultante* soit celle qui vient de la combinaison suivant l'ordre a, d, b, c, e, &c., et ainsi de suite; cela posé, il est clair que toutes ces résultantes

renferment le même nombre de termes, et précisément les mêmes, puisqu'elles renferment tous les termes qui peuvent résulter de la combinaison des n lettres a, b, c, d, e, &c., disposées entre elles de toutes les manières possibles; il ne peut donc y avoir de différence entre deux résultantes, que dans les signes de chacun de leurs termes; or, il est visible que la première résultante donne la seconde, si l'on change dans la première b en c, et réciproquement c en b; mais ce changement augmente ou diminue d'une unité le nombre des variations de chaque terme; d'ou il suit que dans la seconde résultante, tous les termes dont le nombre des variations est impair, auront le signe $+$, et les autres le signe $-$; partant, cette seconde résultante n'est que la première, prise négativement.

"Il est visible pareillement que . . . ." &c.

The proof is thus seen to consist in establishing (1) that the terms of the one "resultant" must, apart from sign, be the same as those of the other; and (2) that the terms of the one resultant are either all affected with the same sign as the like terms of the other, or are all affected with the opposite sign, the comparison of sign being made by comparing the number of variations.

After this, the theorem that when two letters are alike the resultant vanishes is established in a way different from Vandermonde's, but not more satisfactory, viz., by considering what Bézout's rule would lead to in that case.

Application is then made to the problem of elimination, and to the solution of a set of linear simultaneous equations, the mode of treatment being again different from Vandermonde's, but this time with better cause. He says—

"Je suppose maintenant que l'on ait les trois équations

$$0 = {}^1a.\mu + {}^1b.\mu' + {}^1c.\mu'',$$
$$0 = {}^2a.\mu + {}^2b.\mu' + {}^2c.\mu'',$$
$$0 = {}^3a.\mu + {}^3b.\mu' + {}^3c.\mu'',$$

je forme d'abord la résultante des trois lettres a, b, c, suivant l'ordre a, b, c, ce qui donne,

$$ {}^1a.{}^2b.{}^3c - {}^1a.{}^2c.{}^3b + {}^1c.{}^2a.{}^3b - {}^1b.{}^2a.{}^3c + {}^1b.{}^2c.{}^3a - {}^1c.{}^2b.{}^3a. $$

ou

$$ {}^1a.[{}^2b.{}^3c - {}^2c.{}^3b] + {}^2a.[{}^1c.{}^3b - {}^1b.{}^3c] + {}^3a.[{}^1b.{}^2c - {}^1c.{}^2b]; $$

je multiplie ensuite la première des équations précédentes par ${}^2b.{}^3c - {}^2c.{}^3b$, la seconde par ${}^1c.{}^3b - {}^1b.{}^3c$, la troisième par ${}^1b.{}^2c - {}^1c.{}^2b$, et je les ajoute ensemble, ce qui donne,

$$\begin{aligned} 0 = &\ \mu.[{}^1a.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2a.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3a({}^1b.{}^2c - {}^1c.{}^2b)] \\ &+ \mu'.[{}^1b.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2b.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3b({}^1b.{}^2c - {}^1c.{}^2b)] \\ &+ \mu''.[{}^1c.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2c.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3c({}^1b.{}^2c - {}^1c.{}^2b)]\,; \end{aligned}$$

or, il suit de ce que nous venons de voir, que les coëfficiens de μ' et μ'', sont identiquement nuls, puisqu'ils ne sont que la résultante des trois lettres a, b, c, dans laquelle on écrit b, ou c, par-tout où est a; donc, on aura pour l'équation de condition demandée,

$$0 = {}^1a.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2a.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3a.({}^1b.{}^2c - {}^1c.{}^2b)\,;$$

c'est-à-dire, la résultante de la combinaison des trois lettres a, b, c égalée à zéro. On démontreroit la même chose, quel que soit le nombre des équations."

"Pour montrer l'analogie de cette matière, avec l'élimination des équations du premier degré, je suppose que l'on ait les trois équations,

$$\begin{aligned} {}^1p &= {}^1a.\mu + {}^1b.\mu' + {}^1c.\mu'', \\ {}^2p &= {}^2a.\mu + {}^2b.\mu' + {}^2c.\mu'', \\ {}^3p &= {}^3a.\mu + {}^3b.\mu' + {}^3c.\mu''. \end{aligned}$$

Je multiplie, comme ci-devant, la première par $({}^2b.{}^3c - {}^2c.{}^3b)$, la seconde par $({}^1c.{}^3b - {}^1b.{}^3c)$, et la troisième par $({}^1b.{}^2c - {}^1c.{}^2b)$, je les ajoute ensemble, et j'observe que les coëfficiens de μ' et de μ'', sont identiquement nuls dans l'équation qui en résulte; d'où je conclus,

$$\mu = \frac{{}^1p.({}^2b.{}^3c - {}^2c.{}^3b) + {}^2p.({}^1c.{}^3b - {}^1b.{}^3c) + {}^3p.({}^1b.{}^2c - {}^1c.{}^2b)}{{}^1a({}^2b.{}^3c - {}^2c.{}^3b) + {}^2a({}^1c.{}^3b - {}^1b.{}^3c) + {}^3a({}^1b.{}^2c - {}^1c.{}^2b)}\,;$$

on voit donc que le numérateur de l'expression de μ, se forme du dénominateur, en y changeant a en p; on aura ensuite μ' ou μ'', en changeant dans l'expression de μ," &c.

This mode of treatment leaves nothing to be desired. It is that which is most commonly employed in the text-books of the present day.

The next point taken up is the most important in the memoir,

and requires special attention. It is introduced as "a very simple process for considerably abridging the calculation of the equation of condition between a, b, c," &c.—that is to say, the calculation of a resultant. It is, however, something of much more value than this, involving as it does a widely general expansion-theorem to which Laplace's name has been attached, but of which we have already seen special cases stated by Vandermonde. The theorem may be described as giving an expansion of a resultant in the form of an aggregate of terms each of which is a product of resultants of lower degree. Laplace's exposition is as follows:—

"Je suppose que vous ayez deux équations,

$$0 = {}^1a.\mu + {}^1b.\mu' ; \quad 0 = {}^2a.\mu + {}^2b.\mu' ;$$

écrivez $+ab$, et donnez l'indice 1 à la première lettre, et l'indice 2 à la seconde; l'équation de condition demandée sera $+{}^1a.{}^2b - {}^1b.{}^2a = 0$.

"Je suppose que vous ayez trois équations; écrivez $+ab$, combinez ce terme avec la lettre c de toutes les manières possibles, en changeant le signe de chaque terme chaque fois que c change de place, vous aurez ainsi $+abc - acb + cab$; donnez dans chaque terme l'indice 1 à la première lettre, l'indice 2 à la seconde, l'indice 3 à la troisième, et vous aurez $+ {}^1a.{}^2b.{}^3c - {}^1a.{}^2c.{}^3b + {}^1c.{}^2a.{}^3b$; cela posé, au lieu de $+{}^1a.{}^2b.{}^3c$ écrivez $+({}^1a.{}^2b - {}^1b.{}^2a).{}^3c$; au lieu de $+{}^1a.{}^2c.{}^3b$ écrivez $-({}^1a.{}^3b - {}^1b.{}^3a).{}^2c$; et au lieu de $+{}^1c.{}^2a.{}^3b$ écrivez $+({}^2a.{}^3b - {}^2b.{}^3a).{}^1c$; l'équation de condition demandée sera

$$0 = ({}^1a.{}^2b - {}^1b.{}^2a).{}^3c - ({}^1a.{}^3b - {}^1b.{}^3a).{}^2c + ({}^2a.{}^3b - {}^2b.{}^3a).{}^1c.$$

"Je suppose que vous ayez quatre équations, écrivez $+abc - acb + cab$, et combinez ces trois termes avec la lettre d, en observant 1° de n'admettre que les termes dans lesquels c précède d; 2° de changer de signe dans chaque terme toutes les fois que d change de place, et vous aurez

$$+ abcd - acbd + acdb + cabd - cadb + cdab;$$

donnez ensuite l'indice 1 à la première lettre, l'indice 2 à la seconde, &c., et vous aurez

$$+ {}^1a.{}^2b.{}^3c.{}^4d - {}^1a.{}^2c.{}^3b.{}^4d + {}^1a.{}^2c.{}^3d.{}^4b$$
$$+ {}^1c.{}^2a.{}^3b.{}^4d - {}^1c.{}^2a.{}^3d.{}^4b + {}^1c.{}^2d.{}^3a.{}^4b ;$$

cela posé, au lieu de $+{}^1a.{}^2b.{}^3c.{}^4d$ écrivez

$$+({}^1a.{}^2b - {}^1b.{}^2a).({}^3c.{}^4d - {}^3d.{}^4c),$$

et ainsi des autres termes, et l'équation de condition sera

$$\begin{aligned}0 = &({}^1a.{}^2b - {}^1b.{}^2a).({}^3c.{}^4d - {}^3d.{}^4c) - ({}^1a.{}^3b - {}^1b.{}^3a).({}^2c.{}^4d - {}^2d.{}^4c)\\ &+ ({}^1a.{}^4b - {}^1b.{}^4a).({}^2c.{}^3d - {}^2d.{}^3c) + ({}^2a.{}^3b - {}^2b.{}^3a).({}^1c.{}^4d - {}^1d.{}^4c)\\ &- ({}^2a.{}^4b - {}^2b.{}^4a).({}^1c.{}^3d - {}^1d.{}^3c) + ({}^3a.{}^4b - {}^3b.{}^4a).({}^1c.{}^2d - {}^1d.{}^2c).\end{aligned}$$

"Je suppose que vous ayez cinq équations, écrivez les six termes $+abcd - acbd + \ldots$ relatifs à quatre équations, et combinez-les avec la lettre e de toutes les manières possibles, en observant de changer de signe chaque fois que e change de place; donnez ensuite l'indice 1, &c., &c., ... ; au lieu du terme $+{}^1a.{}^2c.{}^3b.{}^4e.{}^5d$ écrivez $({}^1a.{}^3b - {}^1b.{}^3a).({}^2c.{}^5d - {}^2d.{}^5c).{}^4e$, &c. . . . . . .

"Lorsqu'on aura six équations, on combinera les termes $+abcde - abced +$ &c., relatifs à cinq équations avec la lettre f, en observant 1° de n'admettre que les termes dans lesquels e précède f; 2° de changer de signe lorsque f change de place: on transformera ensuite, par la règle précédente, ..."

Notwithstanding the multiplicity of instances, the rule here illustrated is not made altogether clear. This is due to two causes,—first, the linking of one case to the case before it; and, second, the want of explicit notification that the letters $b, d, f \ldots$ are combined in one way, and the intervening letters $c, e, \ldots$ in another. For the sake of additional clearness, let us see all the steps necessary in the case of the resultant of the five equations

$$a_r x_1 + b_r x_2 + c_r x_3 + d_r x_4 + e_r x_5 = 0 \qquad (r = 1, 2, 3, 4, 5),$$

and supposing, as we ought to do, that the case of four equations has not been already dealt with. These steps are—

1°. Combining b with a subject to the condition that a precede b: result—

$$ab.$$

2°. Combining c with this *in every possible way*, the sign being &c.: result—

$$abc - acb + cab.$$

3°. Combining d with each of these terms subject to the condition that c precede d: result—

$$abcd - acbd + acdb + cabd - cadb + cdab.$$

4°. Combining e with each of these terms *in every possible way:* result—

$$\begin{array}{l} abcde - abced + abecd - aebcd + eabcd \\ - acbde + acbed - \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

5°. Appending indices: result—

$$\begin{array}{l} a_1b_2c_3d_4e_5 - a_1b_2c_3e_4d_5 + \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

6°. Changing a_mb_n into $(a_mb_n - b_ma_n)$, c_rd_s into $(c_rd_s - d_rc_s)$, &c.: result—

$$(a_1b_2 - b_1a_2)(c_3d_4 - d_3c_4)e_5 - (a_1b_2 - b_1a_2)(c_3d_5 - d_3c_5)e_4 + \ldots .$$

This is the required resultant in the required form.

It is of the utmost importance to notice that what is accomplished in 1°, 2°, 3°, 4° is simply (α) *the finding of the arrangements of* a, b, c, d, e *subject to the conditions that* a *precede* b, *and* c *precede* d, *and obtaining each arrangement with the sign which it ought to have in accordance with Cramer's rule.* The number of necessary directions might thus be reduced to three, viz., (α), (5), (6), in which case (1), (2), (3), (4) would take their proper places as successive steps of a methodic and expeditious way of accomplishing (α).

Laplace appends a demonstration of the accuracy of this development of the resultant of the nth degree, the line taken being that if the multiplications were performed the terms found would be exactly the 1.2.3.....n terms of the resultant, and would bear the signs proper to them as such.

He then goes on to deal with a rule for obtaining a like development in which as many as possible of the factors of the terms are resultants of the *third* degree.

To do so succinctly he is obliged to introduce a *notation* for resultants. On this point his words are—

"Je désigne par (abc) la quantité

$$abc - acb + cab - bac + bca - cba,$$

et par (ab) la quantité $ab - ba$, et ainsi de suite; par $({}^1a.{}^2b.{}^3c)$ j'indiquerai la quantité (abc), dans les termes de laquelle on donne 1 pour indice à la première lettre, 2 à la seconde, et 3 à la troisième; par $({}^1a.{}^2b)$, je désignerai la quantité (ab) dans les termes de laquelle on donne 1 pour indice à la première lettre, et 2 à la seconde; et ainsi de suite."

We can but remark that here again he leaves little room for improvement: his symbolism is essentially that which is still in common use.

The exposition of the rule is as follows:—

"Je suppose maintenant que vous ayez trois équations, l'équation de condition sera

$$0 = ({}^1a.{}^2b.{}^3c).$$

"Je suppose que vous ayez quatre équations; écrivez $+abc$, et combinez ce terme de toutes les manières possibles avec la lettre d, en observant de changer de signe lorsque d change de place, ce qui donne $+abcd - abdc + adbc - dabc$; donnez l'indice 1 à la première lettre, l'indice 2 à la seconde, &c., et vous aurez

$$+ {}^1a.{}^2b.{}^3c.{}^4d - {}^1a.{}^2b.{}^3d.{}^4c + {}^1a.{}^2d.{}^3b.{}^4c - {}^1d.{}^2a.{}^3b.{}^4c;$$

au lieu du terme $+{}^1a.{}^2b.{}^3c.{}^4d$, écrivez $+({}^1a.{}^2b.{}^3c).{}^4d$; au lieu de $-{}^1a.{}^2b.{}^3d.{}^4c$, écrivez $-({}^1a.{}^2b.{}^4c).{}^3d$, et ainsi de suite, et vous formerez l'équation de condition

$$0 = ({}^1a.{}^2b.{}^3c).{}^4d - ({}^1a.{}^2b.{}^4c).{}^3d + ({}^1a.{}^3b.{}^4c).{}^2d - ({}^2a.{}^3b.{}^4c).{}^1d.$$

"Je suppose que vous ayez cinq équations, combinez les termes $+abcd - abdc +$ &c., relatifs à quatre équations avec la lettre e en observant 1° de n'admettre que les termes dans lesquels d précède e; 2° de changer de signe lorsque e change de place, et vous aurez

$$+ abcde - abdce + abdec + \text{\&c.}$$

donnez l'indice 1 à la première lettre, l'indice 2 à la seconde, &c., et vous aurez

$$+ {}^1a.{}^2b.{}^3c.{}^4d.{}^5e - {}^1a.{}^2b.{}^3d.{}^4c.{}^5e + {}^1a.{}^2b.{}^3d.{}^4e.{}^5c + \text{\&c.};$$

ensuite, au lieu de $+{}^1a.{}^2b.{}^3c.{}^4d.{}^5e$, écrivez $+({}^1a.{}^2b.{}^3c).({}^4d.{}^5e)$; au lieu de $-{}^1a.{}^2b.{}^3d.{}^4c.{}^5e$, écrivez $-({}^1a.{}^2b.{}^4c).({}^3d.{}^5e)$, et ainsi de suite; et en égalant à zero la somme de tous ces termes, vous formerez l'équation de condition demandée.

"Je suppose que vous ayez six équations, combinez les termes $+abcde-$&c., relatifs à cinq équations avec la lettre f, en observant 1° de n'admettre que les termes où e précède f; 2° de changer de signe lorsque f change de place: donnez ensuite 1 pour indice à la première lettre, ...

"Si vous avez sept équations, combinez les termes $+abcdef$ $-$&c. relatifs à six équations avec la lettre g de toutes les manières possibles; pour huit équations, combinez les termes relatifs à sept avec la lettre h, en n'admettant que les termes dans lesquels g précède h, et ainsi du reste."

The really important point in all this is in regard to the manner in which the letters are brought into combination. It will be seen that the set begun with is abc, consequently a precedes b, and b precedes c throughout: then d is combined in every possible way with this: e is combined subject to the condition that d precede e; f is combined subject to the condition that e precede f: g is combined in every way possible: h is combined subject to the condition that g precede h: and so on. It would appear therefore that the letters which are to be combined in every possible way are d and *every third one afterwards*, and that each of the other letters is conditioned to be preceded by the letter which immediately precedes it in the original arrangement $abcdefghi$. . . . Condensing these directions after the manner of the former case, we should draft the rule as follows:—

(*a*) Find every possible arrangement of $abcdefghi$ . . . subject to the conditions that in each arrangement we must have a, b, c in their natural order; d, e, f in their natural order; g, h, i in their natural order; and so on.

(*b*) Prefix to each arrangement its proper sign in accordance with Cramer's rule.

(*c*) Append in order the indices 1, 2, 3 . . . . to the letters of each arrangement.

(*d*) Change $a_m b_n c_r$ into $(a_m.b_n.c_r)$, $d_s e_x f_y$ into $(d_s.e_x f_y)$, &c.

Without saying anything as to the verification of the developments thus obtained, Laplace concludes as follows:—

> "On décomposeroit de la même manière l'équation R en termes composés de facteurs de 4, de 5, &c., dimensions."

To show how this could be effected would have been a tedious matter, if the method of exposition used in the previous cases had been followed, viz., multiplying instances with wearisome iteration of language until the laws for the combination of the letters could with tolerable certainty be guessed. On the other hand, had Laplace condensed his directions in the way we have indicated, the rule for the case in which as many as possible of the factors are of the 4th degree could have been stated as simply as that for either of the two cases he has dealt with. The only changes necessary, in fact, are in parts (1) and (4), and merely amount to writing the letters in consecutive sets of *four* instead of *two* or *three*.

Further, when the rule is condensed in this way, the problem of finding the number of terms in any one of the new developments—a problem which Laplace solves in one case by considering how many terms of the final development each such term gives rise to—is transformed into finding the number of possible arrangements referred to in part (1) of the rule. Where the highest degree of the factors of each term is 2 and the resultant which we wish to develop is of the 2nd degree (which is the case Laplace takes), the number of such arrangements is evidently $(1.2.3 \ldots . n)/(1.2)^s$, s being the highest integer in $n/2$; if the highest degree of the factors is 3, the number of arrangements is

$$\frac{1.2.3 \ldots\ldots n}{(1.2.3)^s\,(1.2)^t},$$

where s is the highest integer in $n/3$ and t the highest integer in $(n-3s)/2$; and so on.

The facts in reduction of the claim which Laplace has to the expansion-theorem now bearing his name are thus seen to be (1) that the case in which as many as possible of the factors of the terms of the expansion are of the 2nd degree had already been given by Vandermonde; (2) that Laplace did not give a statement of his rule in a form suitable for application to all possible cases, and, indeed, was not sufficiently explicit in the statement of it for the first two

cases to enable one readily to see what change would be necessary in applying it to the next case. Notwithstanding these drawbacks, however, there can be no doubt that if any *one* name is to be attached to the theorem it should be that of Laplace.

The sum of his contributions may be put as follows :—

(1) A proof of the theorem regarding the effect of the transposition of two adjacent letters in any of the new functions. (XII. 2)

(2) A mode of arriving at the known solution of a set of simultaneous linear equations. (XIII. 2)

(3) The name *resultant* for the new functions. (XV.)

(4) A notation for a resultant, *e.g.* $({}^1a.{}^2b.{}^3c)$. (VII. 2)

(5) A rule for expressing a resultant as an aggregate of terms composed of factors which are themselves resultants. (XIV. 2)

(6) A mode of finding the number of terms in this aggregate. (XVI.)

LAGRANGE (1773).

[Nouvelle solution du problème du mouvement de rotation d'un corps de figure quelconque qui n'est animé par aucune force accélératrice. *Nouv. Mém. de l'Acad. Roy. .... (de Berlin). Ann.* 1773 (pp. 85–120).]

The position of Lagrange in regard to the advancement of the subject is quite different from that of any of the preceding mathematicians. All of those were explicitly dealing with the problem of elimination, and therefore directly with the functions afterwards known as determinants. Lagrange's work, on the other hand, consists of a number of incidentally obtained algebraical identities which we now-a-days with more or less readiness recognise as relations between functions of the kind referred to, but which unfortunately Lagrange himself did not view in this light, and consequently left behind him as isolated instances. With him x, y, z and x', y', z' and x'', y'', z'' occur primarily as co-ordinates of points in space, and not as coefficients in a triad of linear equations; so that

$$(xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x''),$$

when it does make its appearance, comes as representing six times the bulk of a triangular pyramid and not as the result of an elimination. In days when space of four dimensions was less attempted to be thought about than at present, this circumstance might pos-

sibly account for no advance being made to like identities involving four sets of four letters x, y, z, w; x', y', z', w'; &c.

In this first memoir the algebraical identities are brought together and stated at the outset as follows:—

" LEMME.

" 1. Soient neuf quantités quelconques

$$x,\ y,\ z,\ x',\ y',\ z',\ x'',\ y'',\ z''$$

je dis qu'on aura cette équation identique

$$\begin{aligned}
&(xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x''^2) \\
&\quad = (x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2)(x''^2 + y''^2 + z'')^2 \\
&\quad + 2(xx' + yy' + zz')(xx'' + yy'' + zz'')(x'x'' + y'y'' + z'z'') \\
&\quad - (x^2 + y^2 + z^2)(x'x'' + y'y'' + z'z'')^2 \\
&\quad - (x'^2 + y'^2 + z'^2)(xx'' + yy'' + zz'')^2 \\
&\quad - (x''^2 + y''^2 + z''^2)(xx' + yy' + zz')^2 .
\end{aligned}$$

" Corollaire 1.

" 2. Donc si l'on a entre les neuf quantités précédentes ces six équations

$$\begin{aligned}
x^2 + y^2 + z^2 &= a & x'x'' + y'y'' + z'z'' &= b, \\
x'^2 + y'^2 + z^2 &= a' & xx'' + yy'' + zz'' &= b', \\
x''^2 + y''^2 + z''^2 &= a'' & xx' + yy' + zz' &= b'',
\end{aligned}$$

et qu'on fasse pour abréger

$$\xi = y'z'' - z'y'', \quad \eta = z'x'' - x'z'', \quad \zeta = x'y'' - y'x'',$$

$$\beta = \sqrt{(aa'a'' + 2bb'b'' - ab^2 - a'b'^2 - a''b''^2)};$$

on aura

$$x\xi + y\eta + z\zeta = \beta .$$

On aura de plus les équations identiques suivantes

$$x'\xi + y'\eta + z'\zeta = 0, \quad x''\xi + y''\eta + z''\zeta = 0$$

$$\xi^2 + \eta^2 + \zeta^2 = a'a'' - b^2,$$

$$\begin{aligned}
y'\zeta - z'\eta &= bx' - a'x'', & y''\zeta - z''\eta &= a''x' - bx'', \\
z'\xi - x'\zeta &= by' - a'y'', & z''\xi - x''\zeta &= a''y' - by'', \\
x'\eta - y'\xi &= bz' - a'z'', & x''\eta - y''\xi &= a''z' - bz'',
\end{aligned}$$

qui sont très faciles à vérifier par le calcul.

"Corollaire 2.

"3. Si on prend les trois équations

$$x\xi + y\eta + z\zeta = \beta,$$
$$xx' + yy' + zz' = b'',$$
$$xx'' + yy'' + zz'' = b',$$

et qu'on en tire les valeurs des quantités x, y, z, on aura par les formules connues

$$x = \frac{\beta(y'z'' - z'y'') + b'(\eta z' - \zeta y') + b''(\zeta y'' - \eta z'')}{\xi(y'z'' - z'y'') + \eta(z'x'' - x'z'') + \zeta(x'y'' - y'x'')}$$

$$y = \frac{\beta(z'x'' - x'z'') + b'(\zeta x' - \xi z') + b''(\xi z'' - \zeta x'')}{\xi(y'z'' - z'y'') + \eta(z'x'' - x'z'') + \zeta(x'y'' - y'x'')},$$

$$z = \frac{\beta(x'y'' - y'x'') + b'(y' - \eta x') + b''(\eta x'' - \xi y'')}{\xi(y'z'' - z'y'') + \eta(z'x'' - x'z'') + \zeta(x'y'' - y'x'')};$$

donc faisant les substitutions de l'Art. préc. et supposant pour abréger

$$\alpha = a'a'' - b^2$$

on aura

$$x = \frac{\beta\xi + (a''b'' - bb')x' + (a'b' - bb'')x''}{\alpha},$$

$$y = \frac{\beta\eta + (a''b'' - bb')y' + (a'b' - bb'')y''}{\alpha},$$

$$z = \frac{\beta\zeta + (a''b'' - bb')z' + (a'b' - bb'')z''}{\alpha}."$$

In regard to the first identity here (the so-called lemma), the important and notable point is that the right-hand member is the same kind of function of the nine quantities $x^2+y^2+z^2$, $xx'+yy'+zz'$, $xx''+yy''+zz''$, $xx'+yy'+zz'$, $x'^2+y'^2+z'^2$, $x'x''+y'y''+z'z''$, $xx''+yy''+zz''$, $x'x''+y'y''+z'z''$, $x''^2+y''^2+z''^2$ as the left-hand member is of the nine $x, y, z, x', y', z', x'', y'', z''$. Indeed, without this distinguishing characteristic, the identity would have been to us of comparatively little moment. Possibly Lagrange was aware of it; but, if so, it is remarkable that he did not draw attention to the fact. It is quite true that Lagrange's identity and the modern-looking identity

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}^2 = \begin{vmatrix} x^2+y^2+z^2 & xx'+yy'+zz' & xx''+yy''+zz'' \\ xx'+yy'+zz' & x'^2+y'^2+z'^2 & x'x''+y'y''+z'z'' \\ xx''+yy''+zz'' & x'x''+y'y''+z'z'' & x''^2+y''^2+z''^2 \end{vmatrix}$$

are essentially the same; but no one can deny that the latter contains on the face of it an all-important fact which is hid in the former, and which in Lagrange's time could be made known only by an additional statement in words.

The second identity

$$x'\xi + y'\eta + z'\zeta = 0$$

is a simple case of one of Vandermonde's, viz., that regarding the vanishing of his functions when two of the letters involved were the same.

The third identity

$$\xi^2 + \eta^2 + \zeta^2 = a'a'' - b^2$$

is in modern notation

$$\begin{vmatrix} y' & y'' \\ z' & z'' \end{vmatrix}^2 + \begin{vmatrix} z' & z'' \\ x' & x'' \end{vmatrix}^2 + \begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}^2 = \begin{vmatrix} x'^2 + y'^2 + z'^2 & x'x'' + y'y'' + z'z'' \\ x'x'' + y'y'' + z'z'' & x''^2 + y''^2 + z''^2 \end{vmatrix}$$

and is thus seen to be a simple special instance of a very important theorem afterwards discovered.

The fourth identity

$$y'\zeta - z'\eta = bx' - a'x'',$$

may be expressed in modern notation as follows:—

$$\begin{Vmatrix} y' & z' \\ |z'x''| & |x'y''| \end{Vmatrix} = \begin{vmatrix} x'x'' + y'y'' + z'z'' & x'' \\ x'^2 + y'^2 + z'^2 & x' \end{vmatrix},$$

and, quite probably, has also ere this been generalised in the like notation.

The fifth identity

$$x = \frac{\beta\xi + (a''b'' - bb')x' + (a'b' - bb'')x''}{a},$$

is not so readily transformable, the determinantal theorem which it involves being indeed completely buried. Multiplying both sides by a; then doing away with a, which seems perversely introduced "pour abréger" when no like symbol of abridgment takes the place of $a''b'' - bb'$ or of $a'b' - bb''$; and transposing, we have

$$\beta\xi = x(a'a'' - b^2) - x'(a''b'' - bb') + x''(bb'' - a'b')$$

$$= \begin{vmatrix} x & b'' & b' \\ x' & a' & b \\ x'' & b & a'' \end{vmatrix};$$

that is, finally,

$$|xy'z''|\,.\,|y'z''| = \begin{vmatrix} x & xx'+yy'+zz' & xx''+yy''+zz'' \\ x' & x'^2+y'^2+z'^2 & x'x''+y'y''+z'z'' \\ x'' & x'x''+y'y''+z'z'' & x''^2+y''^2+z''^2 \end{vmatrix},$$

which we recognise as an instance of the multiplication-theorem on putting

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} \times \begin{vmatrix} 1 & x' & x'' \\ 0 & y' & y'' \\ 0 & z' & z'' \end{vmatrix}$$

for the left-hand member.

LAGRANGE (1773).

[Solutions analytiques de quelques problèmes sur les pyramides triangulaires. *Nouv. Mém. de l'Acad. Roy.* . . . . (*de Berlin*) *Ann.* 1773 (pp. 149–176).]

In this memoir also there is a preparatory algebraical portion, the subject being the same as before, and the author's standpoint unchanged. Indeed the two introductions differ only in that the second is a rounding off and slight natural development of the first.

In addition to ξ, η, ζ, we have now ξ', η', ζ', ξ'', η'', ζ'' used as abbreviations for $zy''-yz''$, $xz''-zx''$, ... ; in addition to a, we have a', a'', β, β', β'', standing for $aa''-b'^2$, $aa'-b''^2$, $b'b''-ab$, $bb''-a'b'$, $bb'-a''b''$; and X, Y, Z, X′, Y′, ..., A, A′, . . . are introduced, having the same relation to ξ, η, ζ, ξ', η', ... a, a', . . . as these latter have to x, y, z, x', y', ..., a, a', ... Lagrange then proceeds :—

> "3. Or en substituant les valeurs de ξ, ξ', &c., en x, x', &c., et faisant pour abréger
>
> $$\Delta = xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x'',$$
>
> on trouve
>
> $$\begin{array}{lll} X = \Delta x, & Y = \Delta y, & Z = \Delta z, \\ X' = \Delta x', & Y' = \Delta y', & Z' = \Delta z', \\ X'' = \Delta x'', & Y'' = \Delta y'', & Z'' = \Delta z'', \end{array}$$
>
> donc mettant ces valeurs dans les dernières équations ci-

dessus, on aura en vertu des six équations supposées dans l' Art. 1.

$$\begin{array}{ll} A = \Delta^2 a, & B = \Delta^2 b, \\ A' = \Delta^2 a', & B' = \Delta^2 b', \\ A'' = \Delta^2 a'', & B'' = \Delta^2 b'', \end{array}$$

et de là il est facile de tirer la valeur de Δ^2 en a, a', a'', b, &c. ; car on aura d'abord

$$\Delta^2 = \frac{A}{a} = \frac{a'a'' - \beta^2}{a}$$

et substituant les valeurs de a', a'' et β en a, a', &c. (Art. 1).

$$\Delta^2 = aa'a'' + 2bb'b'' - ab^2 - a'b'^2 - a''b''^2 ;$$

on trouvera la même valeur de Δ^2 par les autres équations. Si on remet dans cette équation les quantités x, y, z, x', &c., on aura la même équation identique que nous avons donnée dans le Lemme ci-dessus (p. 86).

"4. Il est bon de remarquer que la valeur de Δ^2 peut aussi se mettre sous cette forme

$$\Delta^2 = \frac{a\alpha + a'\alpha' + a''\alpha'' + 2(\beta b + \beta' b' + \beta'' b'')}{3} ;$$

or si on multiplie cette équation par Δ^2 et qu'on y substitue ensuite A à la place de $\Delta^2 a$, A' à la place de $\Delta^2 a'$ et ainsi de suite (Art. préc.) on aura

$$\Delta^4 = \frac{A\alpha + A'\alpha' + A''\alpha'' + 2(B\beta + B'\beta' + B''\beta'')}{3} ;$$

ou bien en mettant pour A, A', &c., leurs valeurs en α, α', &c. (Art. 2).

$$\Delta^4 = \alpha\alpha'\alpha'' + 2\beta\beta'\beta'' - \alpha\beta^2 - \alpha'\beta'^2 - \alpha''\beta''^2 ;$$

d'où l'on voit que la quantité Δ^2 et son carré Δ^4 sont des fonctions semblables, l'une de a, a', a'', b, b', b'', l'autre de α, α', α'', β, β', β''.

"5. De plus, comme l'on a (Art. 3)

$$\begin{aligned} & xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'' \\ = & \sqrt{(aa'a'' + 2bb'b'' - ab^2 - a'b'^2 - a''b''^2)} \quad = \Delta , \end{aligned}$$

et qu'il y a entre les quantités x, y, z, x', &c., et a, a', a'', b,

&c., les mêmes relations qu'entre les quantités ξ, η, ζ, ξ', &c., et a, a', a'', β, &c. (Art. 1), on aura donc aussi

$$\begin{aligned}&\xi\eta'\zeta''+\eta\zeta'\xi''+\zeta\xi'\eta''-\xi\zeta'\eta''-\eta\xi'\zeta''-\zeta\eta'\xi''\\ &=\sqrt{(aa'a''+2\beta\beta'\beta''-a\beta^2-a'\beta'^2-a''\beta''^2)}\ =\Delta^2.\end{aligned}$$

Donc on aura cette équation identique et très remarquable

$$\begin{aligned}&\xi\eta'\zeta''+\eta\zeta'\xi''+\zeta\xi'\eta''-\xi\zeta'\eta''-\eta\xi'\zeta''-\zeta\eta'\xi''\\ &=(xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x'')^2."\end{aligned}$$

The remaining portion is of little importance; its main contents are four sets of nine identities each, viz. :—

$$\begin{array}{ll}1.\ x\xi+x'\xi'+x''\xi''=\Delta, & y\xi+y'\xi'+x''\xi''=0,\ \&c.\\ 2.\ x\xi+y\eta+z\zeta=\Delta, & x'\xi+y'\eta+z'\zeta=0,\ \&c.\\ 3.\ \xi=\dfrac{ax+\beta''x'+\beta'x''}{\Delta},\ \&c. & \\ 4.\ x=\dfrac{a\xi+b''\xi'+b'\xi''}{\Delta},\ \&c. & \end{array}$$

Besides the fact that Art. 3 contains a proof of the Lemma of the previous memoir, we have to note the new identity

$$X=\Delta x,$$

which in modern determinantal notation is

$$\begin{Vmatrix}|xz''| & |yx''|\\ |zx'| & |xy'|\end{Vmatrix}=x\,|xy'z''|,$$

—a simple special instance of the theorem regarding what is nowa-days known as "a minor of the determinant adjugate to another determinant."

The last two lines of Art. 4 by implication make it almost certain that Lagrange did not look upon

$$\begin{array}{ll} & xy'z''+yz'x''+zx'y''-xz'y''-yx'z''-zy'x''\\ \text{and} & aa'a''+2bb'b''-ab^2-a'b'^2-a''b''^2\end{array}$$

as functions of the same kind.

The new theorem in Art. 5, which Lagrange justly characterises as "very remarkable," is in modern determinantal notation

$$\begin{Vmatrix}|y'z''| & |z'x''| & |x'y''|\\ |zy''| & |xz''| & |yx''|\\ |yz'| & |zx'| & |xy'|\end{Vmatrix}=\begin{vmatrix}x & y & z\\ x' & y' & z'\\ x'' & y'' & z''\end{vmatrix}^2$$

—a simple instance of the theorem which gives the relation, as we now say, "between a determinant and its adjugate."

In regard to the remaining identities which we have numbered (1), (2), (3), (4), we note that (1) and (3) are not new, although (3) is here given almost in the form desiderated above (pp. 580–1); (2) involves the fact that Δ is the same function of $x, x', x'', y, y', y'', z, z', z''$, as it is of $x, y, z, x', y', z', x'', y'', z''$; and (4) may be transformed as follows:—

$$\begin{aligned} x\Delta &= a\xi + b''\xi' + b'\xi'', \\ &= \begin{vmatrix} a & y & z \\ b'' & y' & z' \\ b' & y'' & z'' \end{vmatrix}, \\ &= \begin{vmatrix} x^2 + y^2 + z^2 & y & z \\ xx' + yy' + zz' & y' & z' \\ xx'' + yy'' + zz'' & y'' & z'' \end{vmatrix}; \end{aligned}$$

so that it may be considered as another disguised instance of the multiplication-theorem, the determinant just reached being equal to

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} \times \begin{vmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{vmatrix}.$$

LAGRANGE (1773).

[Recherches d'Arithmétique. *Nouv. Mém. de l'Acad. Roy. . . . (de Berlin) Ann.* 1773 (pp. 265–312).]

This is an extensive memoir on the numbers "qui peuvent être représentées par la formule $Bt^2 + Ctu + Du^2$". At p. 285 the expression

$$py^2 + 2qyz + rz^2$$

is transformed into

$$Ps^2 + 2Qsx + Rx^2$$

by putting $\qquad y = Ms + Nx,$

and $\qquad z = ms + nx,$

and Lagrange says—

> ". . . . je substitue dans la quantité $PR - Q^2$ les valeurs de P, Q et R, et je trouve en effaçant ce qui se détruit
>
> $$PR - Q^2 = (pr - q^2)\,(Mn - Nm)^2; \; . \; . \; . \; ."$$

which we at once recognise as the simplest case of the theorem connecting (as we now say) the discriminant of any quantic with the discriminant of the result of transforming the quantic by a linear substitution.

Putting now in compact form all the identities obtained from the three preceding memoirs of Lagrange, we have—

(1) $(xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'')^2$
$= aa'a'' + 2bb'b'' - ab^2 - a'b'^2 - a''b''^2,$ (XVII.)
where $a = x^2 + y^2 + z^2, \quad a' = \ldots\ldots$

(2) $\xi^2 + \eta^2 + \zeta^2 = a'a'' - b^2$, where $\xi = y'z'' - z'y''$, $\eta = \ldots\ldots$ (XVIII.)

(3) $y'\zeta - z\eta = bx' - a'x''.$ (XIX.)

(4) $\xi\Delta = \alpha x + \beta''x' + \beta'x''$, where $\alpha = a'a'' - b^2$, $\beta'' = \ldots\ldots$,
and $\Delta = xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x''.$ (XVII. 2)

(5) $X = \Delta x$, where $X = \eta'\zeta'' - \zeta'\eta''.$ (XX.)

(6) $(xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'')^2$
$= \xi\eta'\zeta'' + \eta\zeta'\xi'' + \zeta\xi'\eta'' - \xi\zeta'\eta'' - \eta\xi'\zeta'' - \zeta\eta'\xi''.$ (XXI.)

(7) $PR - Q^2 = (pr - q^2)(Mn - Nm)^2,$ (XXII.)
if $p(Ms + Nx)^2 + 2q(Ms + Nx)(ms + nx) + r(ms + nx)^2$
$= Ps^2 + 2Qsx + Rx^2$ identically.

BÉZOUT (1779).

[Théorie Générale des Equations Algébriques, §§ 195–223, pp. 171–187; §§ 252–270, pp. 208–223. Paris.]

In his extensive treatise on algebraical equations Bézout was bound, as a matter of course, to take up the question of elimination; and, as he had dealt with the subject in a separate memoir in 1764, one might not unreasonably expect to find the treatise giving merely a reproduction of the contents of the memoir in a form suited to a didactic work. Such, however, is far from being the case. He merely mentions the necessary references to the work of Cramer, himself, Vandermonde, and Laplace; and then adds—

"Mais lorsqu'il a été question d'appliquer ces différentes méthodes au problème de l'élimination, envisagé dans toute

son étendue, je me suis bientôt apperçu qu'ils laissoient tous encore beaucoup à desirer du côté de la pratique."

His main objection to the said methods is that when one has to deal with a set of equations of no great generality, with coefficients, it may be, expressed in figures—

"Il faut construire ces formules dans toute la generalité dont les équations sont susceptibles, et faire par conséquent le même travail que si les équations avoient toute cette généralité.

. . . .

(197). Au lieu donc de nous proposer pour but seulement, de donner des formules générales d'élimination dans les équations du premier degré, nous nous proposons de donner une règle qui soit indifféremment et également applicable aux équations prises dans toute leur généralité, et aux équations considerées avec les simplifications qu'elles pourront offrir: une règle dont la marche soit la même pour les unes que pour les autres, mais qui ne fasse calculer que ce qui est absolument indispensable pour avoir la valeur des inconnues que l'on cherche: une règle qui s'applique indifféremment aux équations numériques et aux équations littérales, sans obliger de recourir à aucune formule. Telle est, si je ne me trompe, la règle suivante.

"*Règle générale pour calculer, toutes à la fois, ou séparément, les valeurs des inconnues dans les équations du premier degré, soit littérales soit numériques.*

"(198). Soient u, x, y, z, &c., des inconnues dont le nombre soit n, ainsi que celui des équations.

"Soient a, b, c, d, &c., les coëfficiens respectifs de ces inconnues dans la première équation.

"a', b', c', d', &c., les coëfficiens des mêmes inconnues dans la seconde équation.

"a'', b'', c'', d'', &c., les coëfficiens des mêmes inconnues dans la troisième équation: et ainsi de suite.

"Supposez tacitement que le terme tout connu de chaque équation soit affecté aussi d'une inconnue que je représente par t.

"Formez le produit $uxyzt$ de toutes ces inconnues écrites

dans tel ordre que vous voudrez d'abord ; mais cet ordre une fois admis, conservez-le jusqu'à la fin de l'opération.

" Echangez successivement, chaque inconnue, contre son coëfficient dans la première équation, en observant de changer le signe à chaque échange pair: ce résultat sera, ce que j'appelle, une *première ligne.*

" Echangez dans cette *première ligne*, chaque inconnue, contre son coëfficient dans la seconde équation, en observant, comme ci-devant, de changer le signe à chaque échange pair : et vous aurez une *seconde ligne.*

" Echangez dans cette *seconde ligne*, chaque inconnue, contre son coëfficient dans la troisième équation, en observant de changer le signe à chaque échange pair : et vous aurez une *troisième ligne.*

" Continuez de la même manière jusqu'à la dernière équation inclusivement ; et la dernière *ligne* que vous obtiendrez, vous donnera les valeurs des inconnues de la manière suivante.

" Chaque inconnue aura pour valeur une fraction dont le numérateur sera le coëfficient de cette même inconnue dans la dernière ou n^e *ligne*, et qui aura constamment pour dénominateur le coëfficient que l'inconnue introduite t se trouvera avoir dans cette même n^e *ligne.*"

The application of this very curious rule is illustrated by a considerable number of varied examples, of which we select the second—

" (200). Soient les trois équations suivantes

$$\begin{aligned} ax + by + cz + d &= 0\,, \\ a'x + b'y + c'z + d' &= 0\,, \\ a''x + b''y + c''z + d'' &= 0\,. \end{aligned}$$

" Je les écris ainsi

$$\begin{aligned} ax + by + cz + dt &= 0\,, \\ a'x + b'y + c'z + d't &= 0\,, \\ a''x + b''y + c''z + d''t &= 0\,. \end{aligned}$$

Je forme le produit $xyzt$.

Je change successivement x en a, y en b, z en c, t en d, et observant la règle des signes, j'ai pour première ligne

$$ayzt - bxzt + cxyt - dxyz\,.$$

Je change successivement x en a', y en b', z en c', t en d', et observant la règle des signes, j'ai pour seconde ligne

$$(ab' - a'b)zt - (ac' - a'c)yt + (ad' - a'd)yz$$
$$+ (bc' - b'c)xt - (bd' - b'd)xz + (cd' - c'd)xy.$$

Je change successivement x en a'', y en b'', z en c'', t en d'', et observant la règle des signes j'ai pour troisième ligne

$$\begin{aligned}
&[(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a'']t \\
-&[(ab' - a'b)d'' - (ad' - a'd)b'' + (bd' - b'd)a'']z \\
+&[(ac' - a'c)d'' - (ad' - a'd)c'' + (cd' - c'd)a'']y \\
-&[(bc' - b'c)d'' - (bd' - b'd)c'' + (cd' - c'd)b'']x.
\end{aligned}$$

D'où (198) je tire

$$x = \frac{-[(bc' - b'c)d'' - (bd' - b'd)c'' + (cd' - c'd)b'']}{(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a''},$$

$$y = \frac{+[(ac' - a'c)d'' - (ad' - a'd)c'' + (cd' - c'd)a'']}{(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a'']},$$

$$z = \frac{-[(ab' - a'b)d'' - (ad' - a'd)b'' + (bd' - b'd)a'']}{(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a''}."$$

Among the other examples are included (1) one in which the coefficients in the set of equations are given in figures; (2) one in which some of the coefficients are zero; (3) one showing the simplification possible when the value of only one unknown is wanted; (4) one showing the signification of the vanishing of one of the "*lignes*"; (5) one showing the signification of the absence of one of the unknowns from the last "*ligne*"; and (6) one or two concerned with the allied problem of elimination.

Bézout nowhere gives any reason for his rule; it is used throughout as a pure rule-of-thumb: its effectiveness being manifest, he leaves on the reader the full burden of its arbitrariness. The unreal product $xyzt$ at the very outset must have been a sore puzzle to students, and none the less so because of the certainty which many of them must have felt that a real entity underlay it.

To throw light upon the process, let us compare the above solution of a set of three linear equations with the following solution, which from one point of view may be looked upon as an improvement on the ordinary determinantal modes of solution as presented to modern readers.

The set of equations being

$$\left.\begin{aligned} ax + by + cx + d &= 0 \\ a'x + b'y + c'x + d' &= 0 \\ a''x + b''y + c''x + d'' &= 0 \end{aligned}\right\}$$

we know that the numerator of the values of x, y, z, and the common denominator are

$$-\begin{vmatrix} b & c & d \\ b' & c' & d' \\ b'' & c'' & d'' \end{vmatrix}, \quad +\begin{vmatrix} a & c & d \\ a' & c' & d' \\ a'' & c'' & d'' \end{vmatrix}, \quad -\begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix}, \quad +\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

They are therefore the coefficients of x, y, z, t in the determinant

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ x & y & z & t \end{vmatrix}, \text{ or } \Delta \text{ say.}$$

Thus the problem of solving the set of equations is transformed into finding the development of this determinant. In doing so let us use $[xyz]$ to stand for the determinant of which x, y, z is the last row, and whose other rows are the two rows immediately above x, y, z in Δ: similarly let $[zt]$ stand for the determinant of which z, t is the last row, and its other row the row c'', d'' immediately above z, t in Δ; and so on in all possible cases, including even $[xyzt]$, which of course is Δ itself.

Then clearly we have

$$[xyzt] = a[yzt] - b[xzt] + c[xyt] - d[xyz] \quad . \; . \; . \; . \quad (1)$$

Developing in the same way the four determinants here on the right side, we have as our next step

$$\begin{aligned} [xyzt] = &\quad a(b'[zt] - c'[yt] + d'[yz]) \\ &- b(a'[zt] - c'[xt] + d'[xz]) \\ &+ c(a'[yt] - b'[xt] + d'[xy]) \\ &- d(a'[yz] - b'[xz] + c'[xy]), \\ = &\quad (ab' - a'b)[zt] - (ac' - a'c)[yt] + (ad' - a'd)[yz] \\ &+ (bc' - b'c)[xt] - (bd' - b'd)[xz] + (cd' - c'd)[xy]. \end{aligned}$$

Again, developing the six determinants $[zt]$, $[yt]$, ... in the same way, and rearranging the terms, we have finally

$$\begin{aligned}[xyzt] = &\{(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a''\}t \\ &- \{(ab' - a'b)d'' - (ad' - a'd)b'' + (bd' - b'd)a''\}z \\ &+ \{(ac' - a'c)d'' - (ad' - a'd)c'' + (cd' - c'd)a''\}y \\ &- \{(bc' - b'c)d'' - (bd' - b'd)c'' + (cd' - c'd)b''\}x.\end{aligned}$$

But the coefficients of x, y, z, t in $[x\,y\,z\,t]$ were seen on starting to be the numerators and the common denominator of the values of x, y, z in the given set of equations: hence

$$x = \frac{-\{(bc' - b'c)d'' - (bd' - b'd)c'' + (cd' - c'd)b''\}}{\{(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a''\}}$$

$$y = \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.$$

$$z = \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.$$

Now it is at once manifest that the successive developments here obtained of the determinant $[xyzt]$ are letter by letter identical with the successive "*lignes*" obtained by Bézout from the unreal product $xyzt$; but that instead of having one arbitrary step succeeding another, as in the application of Bézout's rule, there is here a fluent reasonableness characterising the whole process. As for the peculiarities requiring elucidation in the series of special examples

* If the fact at the basis of the process were made use of nowadays, it would be advantageous, of course, in the first instance to simplify the determinant as much as possible. For example, the equations being (Bézout, p. 178)

$$\left.\begin{aligned}2x+4y+5z=22 \\ 3x+5y+2z=30 \\ 5x+6y+4z=43\end{aligned}\right\},$$

we might proceed as follows:—

$$\begin{vmatrix} 2 & 4 & 5 & -22 \\ 3 & 5 & 2 & -30 \\ 5 & 6 & 4 & -43 \\ x & y & z & t \end{vmatrix} = \begin{vmatrix} 0 & 2 & 11 & -6 \\ 1 & 1 & -3 & -8 \\ 0 & -3 & -3 & 9 \\ x & y & z & t \end{vmatrix}$$

$$=3\begin{vmatrix} 0 & 0 & 9 & 0 \\ 1 & 0 & -4 & 5 \\ 0 & -1 & -1 & 3 \\ x & y & z & t \end{vmatrix} = 27\begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 3 \\ x & y & z & t \end{vmatrix}$$

$$=27\{-t+0z-3y-5x\};$$

whence $x=5$, $y=3$, $z=0$.

above referred to, they are seen, when looked at in this light, to be but matters of course.

Not only so, but it will be found that the translation of xy into $[xy]$, &c., is an unfailing key to much that follows in Bézout in connection with the subject. For example, let us take the wide extension of the rule which is expounded later on in the treatise, in a section headed

Considérations utiles pour abréger considérablement le calcul des coëfficients qui servent à l'élimination.

There are in all fifteen pages (pp. 208–223, §§ 252–270) devoted to the subject. The contents of three paragraphs will give a sufficiently clear idea of the nature of the whole. The notation used is identical with that of Laplace, *e.g.*,

$$(ab') = ab' - a'b$$
$$(ab'c'') = (ab' - a'b)c'' - (ab'' - a''b)c' + (a'b'' - a''b')c$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Two of the three selected paragraphs stand as follows :—

"(264.) Cette manière de procéder au calcul des inconnues, en les grouppant, n'est pas applicable seulement à notre object; elle peut en général être appliquée dans toutes les équations du premier degré.

"Si l'on avoit, par exemple, les quatre équations suivantes

$$\begin{aligned} ax + by + cz + dt + e &= 0, \\ a'x + b'y + c'z + d't + e' &= 0, \\ a''x + b''y + c''z + d''t + e'' &= 0, \\ a'''x + b'''y + c'''z + d'''t + e''' &= 0. \end{aligned}$$

En se rappellant que chaque inconnue a pour valeur le coëfficient qu'elle se trouve avoir dans la dernière *ligne*, divisé constamment par celui que l'inconnue introduite aura dans cette même *ligne*, on verra bientôt qu'on peut réduire le calcul à chercher le coëfficient de l'une quelconque des inconnues dans la dernière ligne; parce que de la même manière qu'on en aura calculé un, on calculera de même tous les autres: ou même, lorsqu'on en aura calculé un, on pourra en déduire tous les autres, lorsque les équations auront toute la généralité possible. Or pour avoir la valeur du coëfficient d'une des inconnues dans

la dernière ligne, la question se réduit à calculer la valeur du produit des autres inconnues. Mais pour ne pas se tromper sur les signes, il faudra toujours ne pas perdre de vue, la place que cette inconnue est censée occuper dans le produit de toutes les inconnues. Ainsi, dans le cas présent, au lieu de calculer généralement la dernière *ligne* pour avoir $xyztu$, je calcule seulement cette dernière ligne pour $yztu$: et pour l'avoir de la manière la plus commode, je grouppe en cette manière $yz.tu$, et je procède comme il suit, au calcul des lignes, observant que y est censé à la seconde place.

Première ligne. $-bz.tu-yz.du$,
Seconde ligne. $+(bc').tu-bz.d'u+b'z.du+yz.(de')$,
Troisième ligne. $-(bc').d''u+(bc'').d'u-bz.(d'e'')-(b'c'').du+b'z.(de'')-b''z.(de')$,
Quatrième ligne. $+(bc').(d''e''')-(bc'').(d'e''')+(b'c''').(d'e'')+(b'c'').(de''')$
$-(b'c''').(de'')+(b''c''').(de')$;

c'est le coëfficient de x dans la dernière ligne.

"Pour avoir celui de u, je calculerois de même la valeur de $xyzt$, en le grouppant ainsi, $xy.zt$, et je trouverois pour valeur du coëfficient de u dans la dernière ligne, la quantité

$$(ab').(c''d''')-(ab'').(c'd''')+(ab''').(c'd'')+(a'b'').(cd''')$$
$$-(a'b''').(cd'')+(a''b''').(cd');$$

D'où je conclus

$$x=\frac{+(bc').(d''e''')-(bc'').(d'e''')+(bc''').(d'e'')+(b'c'').(de''')-(b'c''').(de'')+(b''c''').(de')}{(ab').(c''d''')-(ab'').(c'd''')+(ab''').(c'd'')+(a'b'').(cd''')-(a'b''').(cd'')+(a''b''').(cd')}$$

et ainsi de suite.

(265.) Si j'avois les cinq équations suivantes—

$$\begin{aligned}
ax+by+cz+dr+et+f&=0,\\
a'x+b'y+c'z+d'r+e't+f'&=0,\\
a''x+b''y+c''z+d''r+e''t+f''&=0,\\
a'''x+b'''y+c'''z+d'''r+e'''t+f'''&=0,\\
a^{\text{iv}}x+b^{\text{iv}}y+c^{\text{iv}}z+d^{\text{iv}}r+e^{\text{iv}}t+f^{\text{iv}}&=0.
\end{aligned}$$

Je calculerois, par exemple, le coëfficient de x dans la dernière ligne, en calculant $yzr.tu$, ou $yz.rtu$, ou $yz.rt.u$.

Si j'avois six équations dont les inconnues fussent x, y, z, r, s et t, je calculerois, par exemple, le coëfficient de x, en calculant ou $yz.rs.tu$, ou $yzrs.tu$, ou $yzr.stu$, et ainsi de suite.

The next paragraph deals with an illustrative example. The twelve equations—

$$\left.\begin{array}{llllr}
Aa + A'a' + A''a'' & & & & =0 \\
Ab + A'b' + A''b'' & & & & =0 \\
Ac + A'c' + A''c'' & + Ba + B'a' + B''a'' & & & =0 \\
 & + Bb + B'b' + B''b'' & & & =0 \\
 & + Bc + B'c' + B''c'' & & & =0 \\
 & + Bd + B'd' + B''d'' & + Ca + C'a' + C''a'' & & =0 \\
 & & + Cb + C'b' + C''b'' & & =0 \\
 & & + Cc + C'c' + C''c'' & & =0 \\
 & & + Cd + C'd' + C''d'' & + Da + D'a' + D''a'' & =0 \\
 & & & + Db + D'b' + D''b'' & =0 \\
 & & & + Dc + D'c' + D''c'' & =0 \\
Ad + A'd' + A''d'' & & & + Da + D'a' + D''a'' & =0
\end{array}\right\}$$

are given, and what is required is the result of the elimination (*équation de condition*) of the twelve quantities—$A, A', A'', B, B', B'', C, C', C'', D, D', D''$. This is found (the a's in the last equation being misprints for d's) to be—

$$\Gamma(ab'c'').[(bc'd'')^3 - (ab'c'')^2(ab'd'')] = 0.$$

The two paragraphs quoted (§§ 264, 265) show that Bézout could obtain with considerably increased ease and certitude any one of Laplace's expansions of numerator and denominator. What it accomplished in the illustrative example is virtually, in modern symbolism, the reduction of

$$\left|\begin{array}{cccccccccccc}
a & a' & a'' & . & . & . & . & . & . & . & . & . \\
b & b' & b'' & . & . & . & . & . & . & . & . & . \\
c & c' & c'' & a & a' & a'' & . & . & . & . & . & . \\
. & . & . & b & b' & b'' & . & . & . & . & . & . \\
. & . & . & c & c' & c'' & . & . & . & . & . & . \\
. & . & . & d & d' & d'' & a & a' & a'' & . & . & . \\
. & . & . & . & . & . & b & b' & b'' & . & . & . \\
. & . & . & . & . & . & c & c' & c'' & . & . & . \\
. & . & . & . & . & . & d & d' & d'' & a & a' & a'' \\
. & . & . & . & . & . & . & . & . & b & b' & b'' \\
. & . & . & . & . & . & . & . & . & c & c' & c'' \\
d & d' & d'' & . & . & . & . & . & . & d & d' & d''
\end{array}\right|$$

D

to the form

$$|ab'c''| \,.\, |bc'd''|^3 - |ab'c''|^3 \,.\, |ab'd''| \,.$$

Although this can be done nowadays with ease by means of Laplace's expansion-theorem in its modern garb, it may be safely affirmed that Laplace himself, using his own process, would not have succeeded in making the reduction. Considerable importance thus attaches from more than one point of view to Bézout's curious "rule."

The only other section with which we are concerned bears the heading

Méthode pour trouver des fonctions d'un nombre quelconque de quantités, qui soient zéro par ellesmêmes.

In the second paragraph of the section the principle is explained as follows:—

"(216) Concevons un nombre n d'équations du premier degré renfermant un nombre $n+1$ d'inconnues, et sans aucun terme absolument connu.

"Imaginons que l'on augmente le nombre de ces équations de l'une d'entr'elles; alors il est clair que ce que nous appellons la dernière ligne, sera non seulement l'équation de condition nécessaire pour que ce nombre $n+1$ d'équations ait lieu; mais encore que cette équation de condition aura lieu; en sorte qu'elle sera une fonction des coëfficiens de ces équations, laquelle sera zéro par elle-même.

"Voilà donc un moyen très-simple pour trouver un nombre $n+1$* de fonctions d'un nombre $n+1$ de quantités, lesquelles fonctions soient zéro par elles-mêmes."

For example, the pair of equations

$$\left.\begin{array}{l} ax + by + cz = 0 \\ a'x + b'y + c'z = 0 \end{array}\right\}$$

is taken, the first equation is repeated, and for this set of three equations the *équation de condition* is found to be

$$(ab' - a'b)c - (ac' - a'c)b + (bc' - b'c)a = 0 \,.$$

"Or il est clair que la troisième équation n'exprimant rien de différent de la première, cette dernière quantité doit être zéro par elle-même: donc si on a ces deux suites de quantités

* Should be n.

$$\begin{matrix} a, & b, & c \\ a', & b', & c' \end{matrix}$$

on peut être assuré qu'on aura toujours

$$(ab' - a'b)c - (ac' - a'c)b + (bc' - b'c)a = 0\,.$$

"Et si au lieu de joindre la première équation, c'eût été la seconde, nous aurions trouvé de même

$$(ab' - a'b)c' - (ac' - a'c)b' + (bc' - b'c)a' = 0."$$

Similarly in regard to the quantities

$$\begin{matrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{matrix}$$

the identity

$$\begin{aligned} &[(ab' - a'b)\,c'' - (ac' - a'c)\,b'' + (bc' - b'c)\,a'']d \\ -\,&[(ab' - a'b)d'' - (ad' - a'd)b'' + (bd' - b'd)a'']c \\ +\,&[(ac' - a'c)\,d'' - (ad' - a'd)c'' + (cd' - c'd)\,a'']b \\ -\,&[(bc' - b'c)\,d'' - (bd' - b'd)\,c'' + (cd' - c'd)\,b'']a = 0 \end{aligned}$$

and two others are established, the general theorem of course being merely referred to as easily obtainable.

Thus far there is in substance nothing new. What we have obtained is simply a different aspect of Vandermonde's theorem, that *when two indices of either set are alike the function vanishes*, or, as we should now say, *a determinant with two rows identical is equal to zero.* Indeed the identities are used by Vandermonde in Bézout's form when solving a set of simultaneous equations. But what follows is important.

By taking two of these identities

$$\begin{aligned} (ab' - a'b)c\, - (ac' - a'c)b\, + (bc' - b'c)a\, &= 0 \\ (ab' - a'b)c' - (ac' - a'c)b' + (bc' - b'c)a' &= 0\,, \end{aligned}$$

multiplying both sides of the first by d', both sides of the second by d, and subtracting, there is obtained in regard to the quantities

$$\begin{matrix} a, & b, & c, & d \\ a', & b', & c', & d' \end{matrix}$$

the identity

$$(ab' - a'b)(cd' - c'd) - (ac' - a'c)(bd' - b'd) + (bc' - b'c)(ad' - a'd) = 0\,.$$

Similarly by taking the three next identities before obtained, which for shortness we may write in modern notation,

$$|ab'c''|d \;-|ab'd''|c \;+|ac'd''|b \;-|bc'd''|a \;=0\,,$$
$$|ab'c''|d' -|ab'd''|c' +|ac'd''|b' -|bc'd''|a' =0\,,$$
$$|ab'c''|d'' -|ab'd''|c'' +|ac'd''|b'' -|bc'd''|a'' =0\,,$$

there is deduced in regard to the quantities

$$\begin{matrix} a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \end{matrix}$$

the identities

$$|ab'c''|.|de'| - |ab'd''|.|ce'| + |ac'd''|.|be'| - |bc'd''|.|ae'| = 0\,,$$
$$|ab'c''|.|de''| - |ab'd''|.|ce''| + |ac'd''|.|be''| - |bc'd''|.|ae''| = 0\,,$$
$$|ab'c''|.|d'e''| - |ab'd''|.|c'e''| + |ac'd''|.|b'e''| - |bc'd''|.|a'e''| = 0\,.$$

Finally these last three identities are taken, both sides of the first multiplied by f'', both sides of the second by $-f'$, both sides of the third by f, and then by addition there is obtained in regard to the quantities

$$\begin{matrix} a, & b, & c, & d, & e, & f \\ a', & b', & c', & d', & e', & f' \\ a'', & b'', & c'', & d'', & e'', & f'' \end{matrix}$$

the identity

$$|ab'c''|.|d e' f''| - |ab'd''|.|ce'f''| + |ac'd''|.|be'f''| - |bc'd''|.|ae'f''| = 0\,.$$

The subject of what may appropriately be called *vanishing aggregates of determinant-products* is not pursued farther, the concluding paragraph being

> "(223) En voilà assez pour faire connoître la route qu'on doit tenir, pour trouver ces sortes de théorèmes. On voit qu'il y a une infinité d'autres combinaisons à faire, et qui donneront chacune de nouvelles fonctions, qui seront zéro par elles-mêmes: mais cela est facile à trouver actuellement."*

* It is very curious to observe, in passing, that although Bézout does not obtain all his vanishing aggregates directly by means of the principle which he so carefully states at the commencement, nevertheless every one of them can be so obtained. He does not extend the principle beyond the case where only *one* of the original equations is repeated. If, however, we take the equations

$$ax + by + cz + dw = 0\,,$$
$$a'x + b'y + c'z + d'w = 0\,,$$

Our second list of Bézout's contributions thus is:—

(1) An unexplained artificial process for finding the numerators and denominators of fractions which express the values of the unknowns in a set of linear equations, or for finding the resultant of the elimination of n quantities from $n+1$ linear equations,—a process especially useful when the coefficients have particular values. (II. 3 + III. 4 + IV. 2.)

(2) An improved mode of finding Laplace's expansions, especially (but not exclusively) useful when the coefficients have particular values. (XIV. 3.)

(3) A proof of Vandermonde's theorem regarding the effect of the equality of two indices belonging to the same set. (XII. 3.)

(4) A series of identities regarding vanishing aggregates of products. (XXIII.)

HINDENBURG, C. F. (1784).

[*Specimen analyticum de lineis curvis secundi ordinis, in delucidationem Analyseos Finitorum Kaestnerianæ. Auctore Christiano Friderico Rüdigero. Cum praefatione Caroli Friderici Hindenburgii, professoris Lipsiensis.* (xlviii + 74 pp.) pp. xiv–xlviii. *Lipsiæ.*]*

One of the problems dealt with by Rüdiger being the finding of the equation of the conic passing through five given points ("*coefficientium determinatio Traiectoriae secundi ordinis per data quinque puncta*"), Hindenburg, in his preface, takes occasion to show how the generalised problem for $\frac{1}{2}n(n+3)$ points has been treated, pointing out that it is, of course, immediately dependent on the solution of a set of simultaneous linear equations. He directs attention to the labours of Cramer and Bézout, specially lauding the method of the latter, given in the treatise of 1779. Then he

repeat *both* of them so as to have a set of four, and then proceed by the *méthode pour abréger* to find the *équation de condition*, we obtain

$$|ab'|.|cd'| - |ac'|.|bd'| + |ad'|.|bc'| + |bc'|.|ad'| - |bd'|.|ac'| + |cd'|.|ab'| = 0,$$

$$i.e. \quad 2\{|ab'|.|cd'| - |ac'|.|bd'| + |ad'|.|bc'|\} = 0.$$

This is the identity at foot of p. 51, and others are readily seen to be obtainable in the same way.

* My best thanks are due the Committee of Management of University College, London, for the loan of a copy of Hindenburg's tract from the Graves Library.

says—"*Hacc de Opere Bezoldino in universam, quod plurimis adhuc Lectoribus nostris ignotum erit, dicta sufficiant. Nunc Regulam ipsam proponam.*". . . . The seventeen pages which follow, contain a tolerably close Latin translation of the *Règle générale pour calculer* . . . . , and the *Méthode pour trouver* . . . , pp. 172–187, §§ 198–223, which bave been expounded above. Cramer's rule is next given, the second mode of putting it being in words, and the first as follows:—

> "Sint plures Incognitæ z, y, x, w, &c. totidemque Aequationes simplices indeterminatæ
>
> $$\begin{array}{l} A^1 = Z^1z + Y^1y + X^1x + W^1w + \&c. \\ A^2 = Z^2z + Y^2y + X^2x + W^2w + \&c. \\ A^3 = Z^3z + Y^3y + X^3x + W^3w + \&c. \\ A^4 = Z^4z + Y^4y + X^4x + W^4w + \&c. \\ \&c. \quad \&c. \quad \&c. \quad \&c. \quad \&c. \quad \&c. \end{array}$$
>
> Erit, ..., positis terminorum signis, ut praecipitur in fine Tabulæ, pag. seq.
>
> $$z = \frac{\overset{\text{A Y X W V U T} \ldots\ldots}{\text{Permut } (1, 2, 3, 4, 5, 6, 7, \ldots\ldots)}}{\underset{\text{Z Y X W V U T} \ldots\ldots}{\text{Permut } (1, 2, 3, 4, 5, 6, 7, \ldots\ldots)}} \qquad \text{(VII. 3.)}$$"

The similar expressions for y, x, w, v, u, t, are given, and then the "*regula signorum.*" After an illustrative example, the question of the *sequence* of the signs is taken up.

> "Quod si itaque $+sg(1, 2, 3, \ldots, n)$ denotet signorum vicissitudines, quibus hic afficiuntur Permutationum a numeris $1, 2, 3, \ldots n$ singulæ species, et $-sg(1, 2, 3, \ldots n)$ signa *contraria* vel *opposita:* appatet fore
>
> $$\begin{array}{ll} sg(1, 2) & = +sg(1) \quad -sg(1) \\ sg(1, 2, 3) & = +sg(1, 2) \quad -sg(1, 2) \quad +sg(1, 2) \\ sg(1, 2, 3, 4) & = +sg(1, 2, 3) - sg(1, 2, 3) + sg(1, 2, 3) - sg(1, 2, 3) \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$
>
> $$\begin{array}{llll} \text{unde, quia} & sg(1) & \text{est} & +, \text{ facile eruitur} \\ & sg(1, 2) & \text{esse} & + - \\ & sg(1, 2, 3) & \ldots\ldots & + - - + + - \\ & sg(1, 2, 3, 4) & \ldots\ldots & + - - + + - - + + - - + \\ & & & + - - + + - - + + - - + \\ & \ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \end{array}$$

and it is pointed out that the first sign is always +, and the last + or − according as the number $1+2+3+\ldots+(n-1)$ is even or odd.

Bearing in mind that Hindenburg wrote his permutations in a definite order, this remark regarding the sequence of signs entitles us to view him as the author of a combined rule of term-formation and rule of signs, which may be formulated as follows:—

Write the permutations of 1, 2, 3, . . ., *n in ascending order of magnitude as if they were numbers; make the first sign* +, *the second* −, *the next pair contrary in sign to the first pair, the third pair contrary in sign to the second pair, the next six* (1.2.3) *contrary in sign to the first six, the third six contrary in sign to the second six, the fourth six contrary in sign to the third six, the next twenty-four* (1.2.3.4) *contrary in sign to the first twenty-four, and so on.* (II. 4 + III. 5.)

ROTHE, H. A. (1800).

[Ueber Permutationen, in Beziehung auf die Stellen ihrer Elemente. Anwendung der daraus abgeleiteten Sätze auf das Eliminationsproblem. *Sammlung combinatorisch-analytischer Abhandlungen, herausg. v. C. F. Hindenburg*, ii. pp. 263–305.]

Rothe was a follower of Hindenburg, knew Hindenburg's preface to Rüdiger's Specimen Analyticum, and was familiar with what had been done by Cramer and Bézout (see his words at p. 305). His memoir is very explicit and formal, proposition following definition, and corollary following proposition, in the most methodical manner.

The idea which is made the basis of it, that of *place-index* ("Stellenexponent"), is an ill-advised and purposeless modification of Cramer's idea of a "dérangement." The definition is as follows: —In any permutation of the first *n* integers, the *place-index* of any integer is got *by counting the integer itself, and all the elements after it which are less than it.* For example, in the permutation

6, 4, 3, 9, 8, 10, 1, 7, 2, 5

of the first ten integers, the place-index of 9 is 6, and that of 7 is 3. The counting of the integer itself makes the place-index always *one more* than the number of "dérangements" connected with the

integer. This necessitates the introduction of a corresponding modification of Cramer's "rule of signs," viz.

> "3. Willkührlicher Satz. Jede Permutation der Elemente 1, 2, 3, . . . , r, werde mit dem Zeichen + versehen, wenn entweder gar keine, oder eine gerade Menge gerader Zahlen, unter ihren Stellenexponenten vorkommt; mit dem Zeichen − hingegen, wenn die Menge der geraden Zahlen, unter den Stellenexponenten ungerade ist." (III. 6.)

It is difficult to suggest any justification for the changes here introduced. The author himself refers to none. Indeed, in the very next paragraph he points out that to ascertain whether there be an even number of even integers among the place-indices is the same as to diminish each of the place-indices by 1, and ascertain whether there be an even number of odd integers, that is, whether the *sum* of the odd integers be even. He then concludes—

> "Man kann also auch die Regel so ausdrücken: Jede Permutation bekommt das Zeichen + wenn die Summe der um 1 verminderten Stellenexponenten gerade, − hingegen, wenn sie ungerade ist."

This is simply Cramer's rule, and it is the only rule of signs employed henceforward in the memoir, the expression "die Summe der um 1 verminderten Stellenexponenten," occurring over and over again as a periphrasis for "the number of dérangements."

The next four pages are occupied with a very lengthy but thorough investigation of the theorem that *two permutations differ in sign, if they be so related that either is got from the other by the interchange of two of the elements of the latter.* Strictly speaking, however, the proposition proved is something more definite than this, viz.—

If in a permutation of the integers 1.2, . . . r *there be* d *integers intermediate in place and value between any two,* A *and* B, *of the integers, the interchanging of the said two would increase or diminish the number of inversions of order by* $2d+1$. (III. 7.)

The proof consists in finding the sum of the place-indices for the given permutation in terms of d as just defined, c the number of elements less than both A and B and situated between them, f the number of such elements situated to the right of B, and e the

number of elements between A and B in value and situated to the right of B; then finding in like manner the sum of the place-indices for the new permutation; and finally comparing the two sums. The concluding sentence is as follows:—

> "Denn da . . . . , so ist die Summe der Stellenexponenten der zweyten Permutation um $d+e+1-e+d$ oder um $2d+1$ grösser, als bey der ersten Permutation; folglich gilt das auch bey der Summe der um 1 verminderten Stellenexponenten, da bey beyden Permutationen r einerley ist. Also ist die eine Summe gerade, die andere ungerade, folglich haben nach (4) beyde Permutationen verschiedene Zeichen."

As immediate deductions from this, it is pointed out that

The sign of any one permutation may be determined when the sign of any other is known, by counting the number of interchanges necessary to transform the one permutation into the other; (III. 8.)
and that

If one element of a permutation be made to take up a new place, by being, as it were, passed over m *other elements, the sign of the new permutation is the same as, or different from, that of the original according as* m *is even or odd.* (III. 9.)

A third corollary is given, but it is, strictly speaking, a self-evident corollary to the second corollary, and is quite unimportant.

Rothe's next theorem is—

The permutations of 1, 2, 3, ... , n *being arranged after the manner in which numbers are arranged in ascending order of magnitude, any two consecutive permutations will have the same sign, if the first place in which they differ be the* $(4n+3)^{th}$ *or* $(4n+4)^{th}$ *from the end, and will be of opposite sign if the said place be the* $(4n+1)^{th}$ *or* $(4n+2)^{th}$ *from the end.* (III. 10.)

Thus if the permutations of 1, 2, 3, ... , 10 be taken, and arranged as specified, two which will occur consecutively are

$$\begin{array}{l} 8,\ 4,\ 9,\ 3,\ 10,\ 7,\ 6,\ 5,\ 2,\ 1 \\ 8,\ 4,\ 9,\ 5,\ 1,\ 2,\ 3,\ 6,\ 7,\ 10; \end{array}$$

and as the first place in which these differ is the 7^{th} from the end, it is affirmed that the signs preceding them must be alike. The

mode of proving the theorem will be readily understood by seeing it applied to this illustrative example. Taking the permutation

$$8,\ 4,\ 9,\ 3,\ 10,\ 7,\ 6,\ 5,\ 2,\ 1\,,$$

and interchanging 3 and 5 we have the permutation

$$8,\ 4,\ 9,\ 5,\ 10,\ 7,\ 6,\ 3,\ 2,\ 1\,,$$

and thence by cyclical changes the permutation

$$8,\ 4,\ 9,\ 5,\ 1,\ 2,\ 3,\ 6,\ 7,\ 10\,,$$

the number of alterations of sign thus being

$$1+(5+4+3+2+1)$$

$$i.e.\quad 1+\tfrac{1}{2}(5\times 6)\,,$$

—an even number.

Annexed to the theorem is the following corollary, which is not essentially different from Hindenburg's proposition regarding the sequence of signs,—

If the permutations of 1, 2, 3, . . . , n − 1 *be arranged after the manner in which numbers are arranged in ascending order of magnitude, and also in like manner the permutations of* 1, 2, 3, ... , n − 1, n, *then those permutations of the latter arranged set which begin with* r, *say, have in order the same signs as the permutations of the former arranged set, or different signs, according as* r *is odd or even.* (III. 11.)

For example, arranging the permutations of 1, 2, 3, each with its proper sign in front, we have

$$\begin{array}{l} +1,\ 2,\ 3 \\ -1,\ 3,\ 2 \\ -2,\ 1,\ 3 \\ +2,\ 3,\ 1 \\ +3,\ 1,\ 2 \\ -3,\ 2,\ 1\,; \end{array} \qquad \text{(A)}$$

then arranging those permutations of 1, 2, 3, 4 which begin with 3 say, each with its proper sign, we have

$$\begin{array}{l} +3,\ 1,\ 2,\ 4 \\ -3,\ 1,\ 4,\ 2 \\ -3,\ 2,\ 1,\ 4 \\ +3,\ 2,\ 4,\ 1 \\ +3,\ 4,\ 1,\ 2 \\ -3,\ 4,\ 2,\ 1\,; \end{array} \qquad \text{(B)}$$

and the two series of signs are seen to be identical, 3 being an odd number. Viewing this quite independently of the theorem to which it is annexed, it is evident that a change of sign at any point in the series (A) implies a change at the corresponding point in the other series, and consequently attention need only be paid to the first sign of (B) as compared with the first sign of (A). Now the first sign of (A) must necessarily be always plus, there being no inversions; and the first sign of (B) depends on the changes necessary for the transformation of the natural order 1, 2, 3, 4, into 3, 1, 2, 4. The truth of the corollary is thus apparent.

A second corollary is given, but it is of still less consequence, the difference between it and the first being that in the arranged set (B) the place whose occupant remains unchanged may be any one of the n places. (III. 12.)

The next few paragraphs concern the subject of "conjugate permutations" (*verwandte Permutationen*),—apparently a fresh conception. The definition is—

Two permutations of the numbers 1, 2, 3, . . . , n *are called* CONJUGATE *when each number and the number of the place which it occupies in the one permutation are interchanged in the case of the other permutation.* (XXIV.)

For example, the permutations

3, 8, 5, 10, 9, 4, 6, 1, 7, 2 (A)
8, 10, 1, 6, 3, 7, 9, 2, 5, 4 (B)

are conjugate, because 3 is in the 1st place of (A) and 1 is in the 3rd place of (B), 8 is in the 2nd place of (A), and 2 is in the 8th place of B, and so on in every case.

The first theorem obtained is—

Conjugate permutations have the same sign. (III. 13.)

This is proved in a curious and interesting way, a special conjugate pair being considered, viz., the pair just given as an example. To commence with, a square divided into 10×10 equal squares is drawn, the vertical rows of small squares being numbered 1, 2, 3, &c. from left to right, and the horizontal rows 1, 2, 3, &c. from the top downwards. The permutation

3, 8, 5, 10, 9, 4, 6, 1, 7, 2

is then represented by putting a dot in each of the horizontal rows, in the first under 3, in the second under 8, and so on; so that if the rows be taken in order, and the number above each dot read, the given permutation is obtained. For the representation of the conjugate permutation nothing further is necessary: we obtain it at once if we only turn the paper round clockwise until the vertical rows are horizontal, and read off in order the numbers above the dots. In the next place the number of "dérangements" belonging to the permutation 3, 8, 5, ... is indicated by inserting a cross in every small square which is to the left of one dot and above another; thus the two crosses in the first horizontal row correspond to the two "dérangements" 32, 31; the six crosses in the second horizontal row to the six "dérangements" 85, 84, 86, 81, 87, 82; and so on. Then it is observed that if we turn the paper and try to indicate the "dérangements" of the conjugate permutation by inserting a cross in every small square which is to the right of one dot and above another, we obtain exactly the same crosses as before. The signs of the two permutations must thus be alike.

	1	2	3	4	5	6	7	8	9	10
1	×	×	·							
2	×	×		×	×	×	×	·		
3	×	×		×	·					
4	×	×		×		×	×		×	·
5	×	×		×		×	×		·	
6	×	×		·						
7	×	×				·				
8	·									
9		×					·			
10		·								

Immediately following this, the 24 permutations of 1, 2, 3, 4 are given in a column, each one having opposite it, in a parallel column, its conjugate permutation. The existence of *self-conjugate* permutations, *e.g.*, the permutation 3, 4, 1, 2 is thus brought to notice, and the substance of the following theorem in regard to them is given:—

If U_n *be the number of self-conjugate permutations of the first* n *integers, then*

$$U_n = U_{n-1} + (n-1)U_{n-2} \quad . \quad . \quad . \quad . \quad . \quad (\text{xxv.})$$

where $U_1 = 1$ *and* $U_2 = 2$.

This, however, is the only one of his results which Rothe does not attempt to prove.

In the second part of the memoir, which contains the application of the theorems of the first part to the solution of a set of linear

equations, there is not so much that is noteworthy. Methods previously known are followed, the new features being formality and rigour of demonstration.

The coefficients of the equations being

$$\begin{array}{cccccc} 11, & 12, & 13, & \ldots\ldots, & 1r \\ 21, & 22, & 23, & \ldots\ldots, & 2r \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot & \cdot \\ r1, & r2, & r3, & \ldots\ldots, & rr \end{array}$$

it is noted, as Vandermonde had remarked, that the common denominator of the values of the unknown may be got in two ways, viz., by permuting either all the second integers of the couples, 11, 22, 33, ... , *rr*, or all the first integers: but this is supplemented by a proof, that *if any term be taken, e.g.,*

$$16 \,.\, 24 \,.\, 33 \,.\, 47 \,.\, 51 \,.\, 68 \,.\, 79 \,.\, 82 \,.\, 95$$

with the couples so arranged that the first integers are in ascending order, and the sign be determined from the number of inversions in the series of second integers, then the sign obtained will be the same as would be got by arranging the couples so as to have the second integers in ascending order, and determining the sign from the inversions in the series of first integers. The proof rests entirely on the previous theorem, that conjugate permutations have the same sign; indeed the new proposition is little else than another form of this theorem. (III. 14.)

The desirability of an appropriate notation for the cofactor, which any one of the coefficients has in the common denominator, is recognised,* and the want supplied by prefixing f to the coefficient in question; for example, the cofactor of 32 is denoted by

$$\mathfrak{f}\,32.$$

It is thus at once seen that the denominator itself is equal to

$$1n.\mathfrak{f}1n + 2n.\mathfrak{f}2n + \ldots\ldots + rn.\mathfrak{f}rn,$$

or $$n1.\mathfrak{f}n1 + n2.\mathfrak{f}n2 + \ldots\ldots + nr.\mathfrak{f}nr. \qquad \text{(VI. 2.)}$$

Also by this means one of Bézout's (or Vandermonde's) general theorems becomes easily expressible in symbols, viz.,

$$1n.\mathfrak{f}1m + 2n.\mathfrak{f}2m + \ldots\ldots + rn.\mathfrak{f}rm = 0, \qquad \text{(XII. 4.)}$$

* Lagrange's use of a corresponding letter from a different alphabet must not be forgotten.

the proof of which is given as follows. In all the terms of $\mathfrak{f}1m$, every one of the integers except one occurs as the first integer of a couple, and every one of the integers except m occurs as the second integer of a couple: consequently, in every term of $1n\,.\,\mathfrak{f}1m$, the first places of the couples are occupied by the integers from 1 to r inclusive, while in the second places, m is still the only integer awanting, and n occurs twice. Suppose then all the terms of

$$1n.\mathfrak{f}1m + 2n.\mathfrak{f}2m + \ldots\ldots + rn.\mathfrak{f}rm$$

so written, that the first integers of the couples are in ascending order of magnitude, and let us attend to a single term

$$\ldots\ldots\cdot pn\cdot\ldots\ldots\ldots\cdot qn\cdot\ldots\ldots\ldots$$

in which the two couples, having n for second integer, are the p^{th} and q^{th}. If we inquire from which of the expressions $1n.\mathfrak{f}1m$, $2n.\mathfrak{f}2m$, ... this term comes, we see that it is a term of both $pn.\mathfrak{f}pm$ and $qn.\mathfrak{f}qm$, and must, therefore, occur twice. Further, we see that in $pn.\mathfrak{f}qm$ it has the sign of the term

$$\ldots\ldots\cdot pm\cdot\ldots\ldots\ldots\cdot qn\cdot\ldots\ldots\ldots$$

of the common denominator, and that in $qn\,.\,\mathfrak{f}pm$, it has the sign of the term

$$\ldots\ldots\cdot pn\cdot\ldots\ldots\ldots\cdot qm\cdot\ldots\ldots\ldots$$

of the common denominator. But these two terms of the common denominator have different signs: consequently

$$1n.\mathfrak{f}1m + 2n.\mathfrak{f}2m + \ldots\ldots + rn.\mathfrak{f}rm$$

consists of pairs of equal terms with unlike signs, and thus vanishes identically. (XII. 4.)

These preparations having been attended to, the set of r equations with r unknowns is solved by Laplace's method; and a verification made after the manner of Vandermonde. It is also pointed out, that if the solution of a set of equations, say the four

$$\left.\begin{aligned} ax_1 + bx_2 + cx_3 + dx_4 &= s_1 \\ ex_1 + fx_2 + gx_3 + hx_4 &= s_2 \\ ix_1 + kx_2 + lx_3 + mx_4 &= s_3 \\ nx_1 + ox_2 + px_3 + qx_4 &= s_4 \end{aligned}\right\}$$

be

$$\left.\begin{array}{l} x_1 = As_1 + Bs_2 + Cs_3 + Ds_4 \\ x_2 = Es_1 + Fs_2 + Gs_3 + Hs_4 \\ x_3 = Is_1 + Ks_2 + Ls_3 + Ms_4 \\ x_4 = Ns_1 + Os_2 + Ps_3 + Qs_4 \end{array}\right\},$$

then the solution of the set

$$\left.\begin{array}{l} ay_1 + ey_2 + iy_3 + ny_4 = v_1 \\ by_1 + fy_2 + ky_3 + oy_4 = v_2 \\ cy_1 + gy_2 + ly_3 + py_4 = v_3 \\ dy_1 + hy_2 + my_3 + qy_4 = v_4 \end{array}\right\},$$

which has the same coefficients differently disposed, will be

$$\left.\begin{array}{l} y_1 = Av_1 + Ev_2 + Iv_3 + Nv_4 \\ y_2 = Bv_1 + Fv_2 + Kv_3 + Ov_4 \\ y_3 = Cv_1 + Gv_2 + Lv_3 + Pv_4 \\ y_4 = Dv_1 + Hv_2 + Mv_3 + Qv_4 \end{array}\right\}; \qquad \text{(XXVI.)}$$

and hence, that the solution of a set having the special form

$$\left.\begin{array}{l} ax_1 + bx_2 + cx_3 + dx_4 = s_1 \\ bx_1 + ex_2 + fx_3 + gx_4 = s_2 \\ cx_1 + fx_2 + hx_3 + ix_4 = s_3 \\ dx_1 + gx_2 + ix_3 + jx_4 = s_4 \end{array}\right\}$$

will itself take the same form, viz.

$$\left.\begin{array}{l} As_1 + Bs_2 + Cs_3 + Ds_4 = x_1 \\ Bs_1 + Es_2 + Fs_2 + Gs_4 = x_2 \\ Cs_1 + Fs_2 + Hs_3 + Is_4 = x_3 \\ Ds_1 + Gs_2 + Is_3 + Js_4 = x_4 \end{array}\right\}. \qquad \text{(XXVI. 2.)}$$

GAUSS (1801).

[*Disquisitiones Arithmeticæ. Auctore D. Carolo Friderico Gauss.* 167 pp. Lips.]

The connection of Gauss with our theory was very similar to that of Lagrange, and doubtless was due to the fact that Lagrange had preceded him. The fifth chapter of his famous work, which is the only chapter we are concerned with, bears the title "*De formis æquationibusque indeterminatis secundi gradus,*" and its subject may be described in exactly the same words as Lagrange used in

regard to his memoir *Recherches d'Arithmétique* (1773: see above), viz. "les nombres qui peuvent être représentés par la formule $Bt^2 + Ctu + Du^2$."

Gauss writes his form of the second degree thus—

$$axx + 2bxy + cyy;$$

and for shortness speaks of it as the form (a, b, c). The function of the coefficients a, b, c, which was found by Lagrange to be of notable importance in the discussion of the form, Gauss calls the "*determinant* of the form," the exact words of his definition being

> "Numerum $bb - ac$, a cuius indole proprietates formæ (a, b, c) imprimis pendere in sequentibus docebimus, *determinantem* huius formæ uocabimus." (xv. 2.)

Here then we have the first use of the term which with an extended signification has in our day come to be so familiar. It must be carefully noted that the more general functions, to which the name came afterwards to be given, also repeatedly occur in the course of Gauss' work, *e.g.* the function $\alpha\delta - \beta\gamma$ in his statement of Lagrange's theorem (xxii.)

$$b'b' - a'c' = (bb - ac)(\alpha\delta - \beta\gamma)^2.$$

But such functions are not spoken of as belonging to the same category as $bb - ac$. In fact the new term introduced by Gauss was not "determinant" but "determinant of a form," being thus perfectly identical in meaning and usage with the modern term "discriminant."

Notwithstanding the title of the chapter Gauss did not confine himself to forms of two variables. A digression is made for the purpose of considering the ternary quadratic form ("formam ternariam secundi gradus"),

$$axx + a'x'x' + a''x''x'' + 2bx'x'' + 2b'xx'' + 2b''xx',$$

or as he shortly denotes it

$$\begin{pmatrix} a, & a', & a'' \\ b, & b', & b'' \end{pmatrix}.$$

In the matter of nomenclature the following paragraph of this digression is interesting,—

"Ponendo $bb - a'a'' = \mathrm{A}$, $b'b' - aa'' = \mathrm{A}'$, $b''b'' - aa' = \mathrm{A}''$,
$ab - b'b'' = \mathrm{B}$, $a'b' - bb'' = \mathrm{B}'$, $a''b'' - bb' = \mathrm{B}''$,

oritur alia forma

$$\begin{pmatrix} A & A' & A'' \\ B & B' & B'' \end{pmatrix} \quad \ldots . F$$

quam formæ

$$\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix} \quad \ldots . f$$

adjunctam dicemus. Hinc rursus inuenitur, (XXVII.)
denotando breuitatis caussa numerum

$$abb + a'b'b' + a''b''b'' - aa'a'' - 2bb'b'' \quad \text{per } D,$$

$$BB - A'A'' = aD, \quad B'B' - AA'' = a'D, \quad B''B'' - AA' = a''D,$$
$$AB - B'B'' = bD, \quad A'B' - BB'' = b'D, \quad A''B'' - BB' = b''D,$$

unde patet, formæ F adjunctam esse formam

$$\begin{pmatrix} aD, & a'D, & a''D \\ bD, & b'D, & b''D \end{pmatrix}.$$

Numerum D, a cuius indole proprietates formæ ternariæ f imprimis pendent, *determinantem* huius formæ uocabimus; (XV. 2) hoc modo determinans formæ F sit $= DD$, sive æqualis quadrato determinantis formæ f, cui adjuncta est."

In this there is no advance so far as the theory of modern determinants is concerned, the identities given being those numbered (xx) and (xxi) under Lagrange. On the same page, however, an extension is given of Lagrange's theorem (xxii), regarding the determinant of the new form obtained by effecting a linear substitution on a given form. Gauss' words in regard to this are—

"Si forma aliqua ternaria f determinantis D, cuius indeterminatæ sunt x, x', x'' (puta prima $= x$, &c.) in formam ternariam g determinantis E, cuius indeterminatæ sunt y, y', y'', transmutatur per substitutionem talem

$$\begin{aligned} x &= \alpha y + \beta y' + \gamma y'', \\ x' &= \alpha' y + \beta' y' + \gamma' y'', \\ x'' &= \alpha'' y + \beta'' y' + \gamma'' y'', \end{aligned}$$

ubi nouem coefficientes α, β, &c. omnes supponuntur esse numeri integri, breuitatis caussa neglectis indeterminatis simpliciter dicemus, f transire in g per substitutionem (S)

E

$$\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{matrix}$$

atque f *implicare* ipsam g, siue *sub* f *contentam* esse. Ex tali itaque suppositione sponte sequuntur sex equationes pro sex coëfficientibus in g, quas apponere non erit necessarium: hinc autem per calculum facilem sequentes conclusiones euoluuntur:

"I. Designato breuitatis caussa numero

$$\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma''$$

per k inuenitur post debitas reductiones

$$E = kkD, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XXII. } 2.)$$

. . . ."

When freed from its connection with ternary quadratic forms the theorem in determinants here involved is

$$\begin{aligned} \text{If } A_0 &= a_0\alpha_0^2 + a_1\alpha_1^2 + a_2\alpha_2^2 + 2b_0\alpha_1\alpha_2 + 2b_1\alpha_0\alpha_2 + 2b_2\alpha_0\alpha_1, \\ A_1 &= a_0\beta_0^2 + a_1\beta_1^2 + a_2\beta_2^2 + 2b_0\beta_1\beta_2 + 2b_1\beta_0\beta_2 + 2b_2\beta_0\beta_1, \\ A_2 &= a_0\gamma_0^2 + a_1\gamma_1^2 + a_2\gamma_2^2 + 2b_0\gamma_1\gamma_2 + 2b_1\gamma_0\gamma_2 + 2b_2\gamma_0\gamma_1, \\ B_0 &= a_0\beta_0\gamma_0 + a_1\beta_1\gamma_1 + a_2\beta_2\gamma_2 + b_0(\beta_1\gamma_2 + \beta_2\gamma_1) + b_1(\beta_0\gamma_2 + \beta_2\gamma_0) + b_2(\beta_0\gamma_1 + \beta_1\gamma_0) \\ B_1 &= a_0\alpha_0\gamma_0 + a_1\alpha_1\gamma_1 + a_2\alpha_2\gamma_2 + b_0(\alpha_1\gamma_2 + \alpha_2\gamma_1) + b_1(\alpha_0\gamma_2 + \alpha_2\gamma_0) + b_2(\alpha_0\gamma_1 + \alpha_1\gamma_0) \\ B_2 &= a_0\alpha_0\beta_0 + a_1\alpha_1\beta_1 + a_2\alpha_2\beta_2 + b_0(\alpha_1\beta_2 + \alpha_2\beta_1) + b_1(\alpha_0\beta_2 + \alpha_2\beta_0) + b_2(\alpha_0\beta_1 + \alpha_1\beta_0), \end{aligned}$$

then

$$\begin{aligned} &A_0B_0^2 + A_1B_1^2 + A_2B_2^2 - A_0A_1A_2 - 2B_0B_1B_2 \\ &= (a_0b_0^2 + a_1b_1^2 + a_2b_2^2 - a_0a_1a_2 - 2b_0b_1b_2) \\ &\qquad \times (\alpha_0\beta_1\gamma_2 + \beta_0\gamma_1\alpha_2 + \gamma_0\alpha_1\beta_2 - \gamma_0\beta_1\alpha_2 - \alpha_0\gamma_1\beta_2 - \beta_0\alpha_1\gamma_2)^2. \end{aligned}$$

As thus viewed it is an instance of the multiplication-theorem, the product of three determinants (in the modern sense) being expressed as a single determinant.

The multiplication-theorem is also not very distantly connected with the following other statement of Gauss:—

"Si forma ternaria f formam ternarium f' implicat atque haec formam f'': implicabit etiam f ipsam f''. Facillime enim perspicietur, si transeat

f in f' per substitutionem	f' in f'' per substitutionem
$\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{matrix}$	$\begin{matrix} \delta, & \epsilon, & \zeta \\ \delta', & \epsilon', & \zeta' \\ \delta'', & \epsilon'', & \zeta'' \end{matrix}$

f transmutatum iri per substitutionem

$$\begin{array}{lll} \alpha\delta + \beta\delta' + \gamma\delta'', & \alpha\epsilon + \beta\epsilon' + \gamma\epsilon'' & \alpha\zeta + \beta\zeta' + \gamma\zeta'' \\ \alpha'\delta + \beta'\delta' + \gamma'\delta'' & \alpha'\epsilon + \beta'\epsilon' + \gamma'\epsilon'' & \alpha'\zeta + \beta'\zeta' + \gamma'\zeta'' \\ \alpha''\delta + \beta''\delta' + \gamma''\delta'' & \alpha''\epsilon + \beta''\epsilon' + \gamma''\epsilon'' & \alpha''\zeta + \beta''\zeta' + \gamma''\zeta''.\text{"} \end{array} \quad \text{(XVII. 3.)}$$

MONGE (1809).

[Essai d'application de l'analyse a quelques parties de la géométrie élémentaire. *Journ. de l'Ec. Polyt.*, viii. pp. 107–109.]

Lagrange, as we have already seen, was led to certain identities regarding the expression

$$xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x''$$

in the course of investigations on the subject of triangular pyramids. The position of Monge is that of Lagrange reversed. From the theory of equations he derives identities connecting such expressions, and translates them into geometrical theorems.

The simpler of these identities, as being already chronicled, we pass over. At p. 107 he takes the three equations

$$\begin{aligned} a_1u + b_1x + c_1y + d_1z + e_1 &= 0 \\ a_2u + b_2x + c_2y + d_2z + e_2 &= 0 \\ a_3u + b_3x + c_3y + d_3z + e_3 &= 0, \end{aligned}$$

and eliminating every pair of the letters u, x, y, z, obtains the six equations

$$\begin{array}{ll} \beta u + \alpha x + \mathrm{P} = 0 & (1) \\ \gamma x + \beta y + \mathrm{Q} = 0 & (2) \\ \delta y + \gamma z + \mathrm{M} = 0 & (3) \\ \alpha z + \delta u + \mathrm{N} = 0 & (4) \\ \gamma u - \alpha y + \mathrm{S} = 0 & (5) \\ \beta z - \delta x + \mathrm{R} = 0 & (6); \end{array}$$

the ten letters

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \mathrm{M},\ \mathrm{N},\ \mathrm{P},\ \mathrm{Q},\ \mathrm{R},\ \mathrm{S}$$

being used to stand for the lengthy expressions which we nowadays denote by

$$|b_1c_2d_3|,\ |a_1c_2d_3|,\ |a_1b_2d_3|,\ |a_1b_2c_3|,$$
$$a_1b_2e_3|,\ |b_1c_2e_3\ |,\ |c_1d_2e_3|,\ -|a_1d_2e_3|,\ |a_1c_2e_3|,\ |b_1d_2e_3|.$$

Then, taking triads of these six equations, *e.g.*, the triads (1), (2), (5), he derives the identities

$$\left.\begin{aligned} \alpha Q + \beta S - \gamma P &= 0 \\ \delta P + \alpha R - \beta N &= 0 \\ -\gamma N + \delta S + \alpha M &= 0 \\ -\beta M + \gamma R + \delta Q &= 0 \end{aligned}\right\},$$

or

$$\left.\begin{aligned} -|b_1c_2d_3|.|a_1d_2e_3| + |a_1c_2d_3|.|b_1d_2e_3| - |a_1b_2d_3|.|c_1d_2e_3| &= 0 \\ |a_1b_2c_3|.|c_1d_2e_3| + |b_1c_2d_3|.|a_1c_2e_3| - |a_1c_2d_3|.|b_1c_2e_3| &= 0 \\ -|a_1b_2d_3|.|b_1c_3e_3| + |a_1b_2c_3|.|b_1d_2e_3| + |b_1c_2d_3|.|a_1b_2e_3| &= 0 \\ -|a_1c_2d_3|.|a_1b_2e_3| + |a_1b_2d_3|.|a_1c_2e_3| - |a_1b_2c_3|.|a_1d_2e_3| &= 0 \end{aligned}\right\} \text{(xxiii. 2.)}$$

which in their turn, he says, by processes of elimination, may be the source of many others. For example, each of the four being linear and homogeneous in α, β, γ, δ, these letters may all be eliminated with the result

$$RS + QN - PM = 0,$$

or

$$|a_1c_2e_3|.|b_1d_2e_3| - |a_1d_2e_3|.|b_1c_2e_3| - |c_1d_2e_3|.|a_1b_2e_3| = 0.$$

Also, eliminating P from the first and second, S from the first and third, Q from the first and fourth, and so on, we have

$$\begin{aligned} -\beta\gamma N + \delta\alpha Q + \beta\delta S + \alpha\gamma R &= 0, \\ \alpha\beta M + \gamma\delta P - \beta\gamma N - \delta\alpha Q &= 0, \\ \alpha\beta M - \gamma\delta P + \beta\delta S - \alpha\gamma R &= 0, \\ \&c. \qquad \&c. & \end{aligned}$$

i.e.

$$\left.\begin{aligned} -|a_1c_2d_3|.|a_1b_2d_3|.|b_1c_2e_3| - |a_1b_2c_3|.|b_1c_2d_3|.|a_1d_2e_3| \\ +|a_1c_2d_3|.|a_1b_2c_3|.|b_1d_2e_3| + |b_1c_2d_3|.|a_1b_2d_3|.|a_1c_2e_3| \end{aligned}\right\} = 0,$$

$$\&c. \qquad \&c. \qquad \text{(xxviii.)}$$

Monge does not pursue the subject further. His method, however, is seen to be quite general; and we can readily believe that he possessed numerous other identities of the same kind. This is borne out by a statement in Binet's important memoir of 1812. Binet, who was familiar with what had been done by Vandermonde, Laplace, and Gauss, says (p. 286):—"M. Monge m'a communiqué, depuis la lecture de ce mémoire, d'autres théorèmes très-remarquables sur ces résultantes; mais ils ne sont pas du genre de ceux que nous nous proposons de donner ici."

HIRSCH (1809).

[Sammlung von Aufgaben aus der Theorie der algebraischen Gleichungen, von Meier Hirsch. pp. 103–107. Berlin, 1809.]

The 4th Chapter *Von der Elimination u. s. w.*, contains five pages on the subject of the solution of simultaneous linear equations. These embrace nothing more noteworthy than a statement, without proof, of Cramer's rule, separated into three parts (iv., iii. 2, v.), and carefully worded.

BINET (May 1811).

[Mémoire sur la théorie des axes conjugués et des momens d'inertie des corps. *Journ. de l'École Polytechnique*, ix. (pp. 41–67), pp. 45, 46.]*

In this well-known memoir, in which the conception of the *moment of inertia of a body with respect to a plane* was first made known, there repeatedly occur expressions, which at the present day would appear in the notation of determinants. There is only one paragraph, however, containing anything new in regard to these functions. It stands as follows:—

"Le moment d'inertie minimum pris par rapport au plan (C) a pour valeur

$$\Sigma mk^2 = f^2 \times$$

$$\frac{ABC - AF^2 - BE^2 - CD^2 + 2DEF}{g^2(BC - F^2) + h^2(AC - E^2) + i^2(AB - D^2) + 2gh(EF - CD) + 2gi(DF - BE) + 2hi(DE - AF)}.$$

Si, dans le numérateur,

$$ABC - AF^2 - BE^2 - CD^2 + 2DEF$$

on remplace A, B, C, &c. par Σmx^2, Σmy^2, &c. que ces lettres représentent, on a

$$\Sigma mx^2 \Sigma my^2 \Sigma mz^2 - \Sigma mx^2(\Sigma myz)^2 - \Sigma my^2(\Sigma mxz)^2$$
$$- \Sigma mz^2(\Sigma mxy)^2 + 2\Sigma mxy \Sigma mxz \Sigma myz,$$

et l'on peut s'assurer que cette expression est identique à

$$\Sigma mm'm''(xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'')^2;$$

par une transformation analogue, on peut ramener la quantité

* An abstract of this is given in the *Nouv. Bull. des Sciences par la Société Philomatique*, ii. pp. 312–316.

$$g^2(\mathrm{BC}-\mathrm{F}^2) + h^2(\mathrm{AC}-\mathrm{E}^2) + i^2(\mathrm{AB}-\mathrm{D}^2)$$
$$+ 2gh(\mathrm{EF}-\mathrm{CD}) + 2gi(\mathrm{DF}-\mathrm{BE}) + 2hi(\mathrm{DE}-\mathrm{AF}),$$

à celle-ci

$$\Sigma mm'[g(yz'-zy') + h(zx'-xz') + i(xy'-yx')]^2."$$

Now the numerator referred to would at the present day be written.

$$\begin{vmatrix} \mathrm{A} & \mathrm{D} & \mathrm{E} \\ \mathrm{D} & \mathrm{B} & \mathrm{F} \\ \mathrm{E} & \mathrm{F} & \mathrm{C} \end{vmatrix},$$

and since Σmx^2, &c. stand for $mx^2 + m_1x_1^2 + m_2x_2^2 + \ldots$, &c., the first identity given may be put in the form

$$\begin{vmatrix} mx^2+m_1x_1^2+m_2x_2^2+\ldots & mxy+m_1x_1y_1+m_2x_2y_2+\ldots & mxz+m_1x_1z_1+m_2x_2z_2+\ldots \\ mxy+m_1x_1y_1+m_2x_2y_2+\ldots & my^2+m_1y_1^2+m_2y_2^2+\ldots & myz+m_1y_1z_1+m_2y_2z_2+\ldots \\ mxz+m_1x_1z_1+m_2x_2z_2+\ldots & myz+m_1y_1z_1+m_2y_2z_2+\ldots & mz^2+m_1z_1^2+m_2z_2^2+\ldots \end{vmatrix}$$

$$= mm_1m_2\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{vmatrix}^2 + mm_1m_3\begin{vmatrix} x & x_1 & x_3 \\ y & y_1 & y_3 \\ z & z_1 & z_3 \end{vmatrix}^2 + \ldots \quad (\text{XVIII. } 2.)$$

where $x_1, y_2, \ldots$ are for convenience written instead of $x', y'', \ldots$ It will be seen that this is an important extension of a theorem of Lagrange, the latter theorem being the very special case of the present obtained by putting $m = m_1 = m_2 = 1$, and $m_3 = m_4 = \ldots = 0$, —a fact which is brought still more clearly into evidence if, instead of the left-hand member of the identity, we write the modern contraction for it, viz.

$$\begin{vmatrix} mx & m_1x_1 & m_2x_2 & m_3x_3 & \ldots \\ my & m_1y_1 & m_2y_2 & m_3y_3 & \ldots \\ mz & m_1z_1 & m_2z_2 & m_3z_3 & \ldots \end{vmatrix} \times \begin{vmatrix} x & x_1 & x_2 & x_3 & \ldots \\ y & y_1 & y_2 & y_3 & \ldots \\ z & z_1 & z_2 & z_3 & \ldots \end{vmatrix}.$$

Again the denominator

$$g^2(\mathrm{BC}-\mathrm{F}^2) + h^2(\mathrm{AC}-\mathrm{E}^2) + i^2(\mathrm{AB}-\mathrm{D}^2)$$
$$+ 2gh(\mathrm{EF}-\mathrm{CD}) + 2gi(\mathrm{DF}-\mathrm{BE}) + 2hi(\mathrm{DE}-\mathrm{AF})$$

being in modern notation

$$\begin{vmatrix} \cdot & g & h & i \\ g & \mathrm{A} & \mathrm{D} & \mathrm{E} \\ h & \mathrm{D} & \mathrm{B} & \mathrm{F} \\ i & \mathrm{E} & \mathrm{F} & \mathrm{C} \end{vmatrix}$$

the second identity may be written

$$\begin{vmatrix} \cdot & g & h & i \\ g & mx^2+m_1x_1{}^2+\dots & mxy+m_1x_1y_1+\dots & mxz+m_1x_1z_1+\dots \\ h & mxy+m_1x_1y_1+\dots & my^2+m_1y_1{}^2+\dots & myz+m_1y_1z+\dots \\ i & mxz+m_1x_1z_1+\dots & myz+m_1y_1z_1+\dots & mz^2+m_1z_1{}^2+\dots \end{vmatrix}$$

$$= mm_1\begin{vmatrix} g & x & x_1 \\ h & y & y_1 \\ i & z & z_1 \end{vmatrix}^2 + mm_2\begin{vmatrix} g & x & x_2 \\ h & y & y_2 \\ i & z & z_2 \end{vmatrix}^2 + m_1m_2\begin{vmatrix} g & x_1 & x_2 \\ h & y_1 & y_2 \\ i & z_1 & z_2 \end{vmatrix} + \dots \quad \text{(XXIX.)}$$

This also is an important theorem, and is not so much an extension of previous work as a breaking of fresh ground.

BINET (November 1811).

[Sur quelques formules d'algèbre, et sur leur application à des expressions qui ont rapport aux axes conjugués des corps. *Nouv. Bull. des Sciences par la Société Philomatique*, ii. pp. 389–392.]

In this paper Binet returns to the consideration of the first of the two identities which have just been referred to, writing it now in the form

$$\begin{aligned}&\Sigma(xy'z''-xz'y''+yz'x''-yx'z''+zx'y''-zy'x'')^2 \\ &=\Sigma x^2\Sigma y^2\Sigma z^2-\Sigma x^2(\Sigma yz)^2-\Sigma y^2(\Sigma xz)^2-\Sigma z^2(\Sigma xy)^2+2\Sigma xy\Sigma xz\Sigma yz.\end{aligned}$$

He puts it in the same category as the identity

$$\Sigma(y'z-zy')^2=\Sigma y^2\Sigma z^2-(\Sigma yz)^2,$$

which he speaks of as being then known. Further, he says

"Ces deux formules sont du même genre que la suivante

$$\Sigma\begin{Bmatrix} ux'y''z'''-ux'z''y'''+uy'z''x'''-uy'x''z'''+uz'x''y'''-uz'y''x'''+xy'u''z'''-xy'z''u''' \\ +xz'y''u'''-xz'u''y'''+xu'z''y'''-xu'y''z'''+yz'u''x'''-yz'x''u'''+yu'x''z'''-yu'z''x''' \\ +yx'z''u'''-yx'u''z'''+zu'y''x'''-zu'x''y'''+zx'y''u'''-zx'u''y'''+zy'x''u'''-zy'u''x''' \end{Bmatrix}^2$$

$$\begin{aligned}&=\Sigma u^2\Sigma x^2\Sigma y^2\Sigma z^2-\Sigma u^2\Sigma x^2(\Sigma yz)^2-\Sigma u^2\Sigma y^2(\Sigma xz)^2-\Sigma u^2\Sigma z^2(\Sigma xy)^2 \\ &-\Sigma x^2\Sigma y^2(\Sigma uz)^2-\Sigma x^2\Sigma z^2(\Sigma uy)^2-\Sigma y^2\Sigma z^2(\Sigma ux^2) \\ &+2\Sigma u^2\Sigma xy\Sigma xz\Sigma yz+2\Sigma x^2\Sigma uy\Sigma uz\Sigma yz+2\Sigma y^2\Sigma ux\Sigma uz\Sigma xz \\ &+2\Sigma z^2\Sigma ux\Sigma uy\Sigma xy+(\Sigma ux)^2(\Sigma yz)^2+(\Sigma uy)^2(\Sigma xz)^2+(\Sigma uz)^2(\Sigma xy)^2 \\ &-2\Sigma ux\Sigma xy\Sigma yz\Sigma zu-2\Sigma uy\Sigma yz\Sigma zx\Sigma xu-2\Sigma uy\Sigma yx\Sigma xz\Sigma zu,\end{aligned}$$

—a result which in modern notation would take the form

$$\begin{vmatrix} u & u_1 & u_2 & u_3 \\ x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \end{vmatrix}^2 + \begin{vmatrix} u & u_1 & u_2 & u_4 \\ x & x_1 & x_2 & x_4 \\ y & y_1 & y_2 & y_4 \\ z & z_1 & z_2 & z_4 \end{vmatrix}^2 + \ldots\ldots$$

$$= \begin{vmatrix} u^2+u_1^2+\ldots & ux+u_1x_1+\ldots & uy+u_1y_1+\ldots & uz+u_1z_1+\ldots \\ ux+u_1x_1+\ldots & x^2+x_1^2+\ldots & xy+x_1y_1+\ldots & xz+x_1z_1+\ldots \\ uy+u_1y_1+\ldots & xy+x_1y_1+\ldots & y^2+y_1^2+\ldots & yz+y_1z_1+\ldots \\ uz+u_1z_1+\ldots & xz+x_1z_1+\ldots & yz+y_1z_1+\ldots & z^2+z_1^2+\ldots \end{vmatrix} . \qquad \text{(XVIII. 3.)}$$

It is thus clear that, in November 1811, Binet was well on the way towards a great generalisation. He even says that the three identities may be looked upon

> "comme les trois premières d'une suite de formules construites d'après une même loi facile à saisir."

He merely indicates, however, the mode of proof he would adopt for the results obtained, and refers to possible applications of them in investigations regarding the Method of Least Squares (Laplace, *Connaissance des Tems*, 1813) and the Centre of Gravity (Lagrange, *Mém. de Berlin*, 1783). The mode of proof need not be given here, as it turns up again in the far more important memoir in which the theorem in all its generality falls to be considered.

DE PRASSE (1811).

[Commentationes Mathematicæ. Auctore Mauricio de Prasse. 120 pp. Lips., 1804, 1812. Pp. 89–102; Commentatio vii.*: Demonstratio eliminationis Cramerianæ.]

Of previous writings the one which De Prasse's most resembles is Rothe's. There is less of it, and it shows less freshness; but there is the same stiff formality of arrangement, and the same effort at rigour of demonstration.

* Separate copies of the *Demonstratio eliminationis Cramerianæ* are also to be found, bearing the invitation title-page:

> *Ad memoriam Kregelio-Sternbachianam in auditorio philosophorum die* xviii *Julii* MDCCCXI. *h.* ix *celebrandam invitant ordinum Academiæ Lips. Decani seniores cœterique adsessores . . . . Demonstratio eliminationis Cramerianæ.*

It is these copies which fix the date. See *Nature*, xxxvii. pp. 246, 247.

The definition of a permutation (*variatio*) being given, the first problem (which, however, is called a theorem) is propounded, viz., to tabulate the permutations of α, β, γ, δ, . . . (" *Variationum ex elementis* α, β, γ, . . . *constructarum et in Classes combinatorias digestarum Tabulam parare*"). The result is

α	β	γ	δ
	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$
	$\beta\alpha$	$\beta\gamma$	$\beta\delta$
	$\gamma\alpha$	$\gamma\beta$	$\gamma\delta$
	$\delta\alpha$	$\delta\beta$	$\delta\gamma$
		$\alpha\beta\gamma$	$\alpha\beta\delta$
		$\alpha\gamma\beta$	$\alpha\gamma\delta$
		$\alpha\delta\beta$	$\alpha\delta\gamma$
		$\beta\alpha\gamma$	$\beta\alpha\delta$
		$\beta\gamma\alpha$	$\beta\gamma\delta$
		$\beta\delta\alpha$	$\beta\delta\gamma$
		$\gamma\alpha\beta$	$\gamma\alpha\delta$
		$\gamma\beta\alpha$	$\gamma\beta\delta$
		$\gamma\delta\alpha$	$\gamma\delta\beta$
		$\delta\alpha\beta$	$\delta\alpha\gamma$
		$\delta\beta\alpha$	$\delta\beta\gamma$
		$\delta\gamma\alpha$	$\delta\gamma\beta$
			$\alpha\beta\gamma\delta$
			$\alpha\beta\delta\gamma$
			$\alpha\gamma\beta\delta$
			$\alpha\gamma\delta\beta$
			$\alpha\delta\beta\gamma$
			$\alpha\delta\gamma\beta$
			$\beta\alpha\gamma\delta$
			$\beta\alpha\delta\gamma$
			$\beta\gamma\alpha\delta$
			$\beta\gamma\delta\alpha$
			$\beta\delta\alpha\gamma$
			$\beta\delta\gamma\alpha$
			$\gamma\alpha\beta\delta$
			$\gamma\alpha\delta\beta$
			$\gamma\beta\alpha\delta$
			$\gamma\beta\delta\alpha$
			$\gamma\delta\alpha\beta$
			$\gamma\delta\beta\alpha$
			$\delta\alpha\beta\gamma$
			$\delta\alpha\gamma\beta$
			$\delta\beta\alpha\gamma$
			$\delta\beta\gamma\alpha$
			$\delta\gamma\alpha\beta$
			$\delta\gamma\beta\alpha$

The first row of the permutations involving two letters is got by taking the first letter of the previous row and annexing each of the others to it in succession and in the order of their occurrence; the second row is got in like manner from the second letter; and so on. Similarly the first row of permutations involving three letters is got from $\alpha\beta$ the first obtained permutation of two letters, the second row from $\alpha\gamma$ the next obtained permutation of two letters, and so on.*

The second problem (and on this occasion actually so designated) is somewhat quaint in its indefiniteness, viz., to prefix to each permutation the sign + or the sign −, so that the sum of all the permutations involving the same number of letters (>1) may vanish ("*Singulis Variationibus, omissis repetitionibus, signa + et − ita praefigere, ut summa secundæ et cujuslibet classis insequentis evanescat*"). There is no indefiniteness or multiplicity about the solution, which in substance is:—Make the permutations in every row of the preceding table alternately + and −, the first sign of all being +, and the first permutation of every other row having the same sign as the permutation from which it was derived. In this way the table becomes

$$\left.\begin{array}{cccc} +\alpha, & -\beta, & +\gamma, & -\delta \end{array}\right\}$$

$$\left.\begin{array}{ccc} +\alpha\beta, & -\alpha\gamma, & +\alpha\delta \\ -\beta\alpha, & +\beta\gamma, & -\beta\delta \\ +\gamma\alpha, & -\gamma\beta, & +\gamma\delta \\ -\delta\alpha, & +\delta\beta, & -\delta\gamma \end{array}\right\}$$

$$\left.\begin{array}{cc} +\alpha\beta\gamma, & -\alpha\beta\delta \\ -\alpha\gamma\beta, & +\alpha\gamma\delta \\ +\alpha\delta\beta, & -\alpha\delta\gamma \\ -\beta\alpha\gamma, & +\beta\alpha\delta \\ +\beta\gamma\alpha, & -\beta\gamma\delta \\ -\beta\delta\alpha, & +\beta\delta\gamma \\ +\gamma\alpha\beta, & -\gamma\alpha\delta \\ -\gamma\beta\alpha, & +\gamma\beta\delta \\ +\gamma\delta\alpha, & -\gamma\delta\beta \\ -\delta\alpha\beta, & +\delta\alpha\gamma \\ +\delta\beta\alpha, & -\delta\beta\gamma \\ -\delta\gamma\alpha, & +\delta\gamma\beta \end{array}\right\}$$

* It will be seen that the order in which the permutations come to hand in this process of tabulation is the order in which they would be arranged according to magnitude if each permutation were viewed as a number of which α, β, γ, δ were the digits, α being $<\beta<\gamma<\delta$ ("ordo lexicographicus," "lexicographische Anordnung" of Hindenburg).

$$\left.\begin{array}{l} +\alpha\beta\gamma\delta \\ -\alpha\beta\delta\gamma \\ -\alpha\gamma\beta\delta \\ +\alpha\gamma\delta\beta \\ +\alpha\delta\beta\gamma \\ -\alpha\delta\gamma\beta \\ \\ -\beta\alpha\gamma\delta \\ +\beta\alpha\delta\gamma \\ +\beta\gamma\alpha\delta \\ -\beta\gamma\delta\alpha \\ -\beta\delta\alpha\gamma \\ +\beta\delta\gamma\alpha \\ \\ +\gamma\alpha\beta\delta \\ -\gamma\alpha\delta\beta \\ -\gamma\beta\alpha\delta \\ +\gamma\beta\delta\alpha \\ +\gamma\delta\alpha\beta \\ -\gamma\delta\beta\alpha \\ \\ -\delta\alpha\beta\gamma \\ +\delta\alpha\gamma\beta \\ +\delta\beta\alpha\gamma \\ -\delta\beta\gamma\alpha \\ -\delta\gamma\alpha\beta \\ +\delta\gamma\beta\alpha \end{array}\right\} .$$

A proof by the method of mathematical induction (so-called) is given that with these signs the sum of all the permutations of any group vanishes.

Up to this point the essence of what has been furnished is a combined rule of term-formation and rule of signs. (II. 5 + III. 15.) In connection with it Bézout's rule of the year 1764 may be recalled.

The third problem is to determine the sign of any single permutation from consideration of the permutation itself. The solution is:—Under each letter of the given permutation put all the letters which precede it in the natural arrangement and which are not found to precede it in the given permutation; and make the sum + or − according as the total number of such letters is even or odd.

"EXEMP. Datæ complexiones sint hæ:

$$\epsilon\gamma\delta\beta, \quad \delta\alpha\epsilon\gamma, \quad \epsilon\delta\gamma\alpha, \quad \delta\beta\epsilon\gamma.$$

Literæ secundum I subjiciantur

$\alpha\,\alpha\,\alpha\,\alpha$	$\alpha\,.\,\beta\beta$	$\alpha\,\alpha\,\alpha\,.$	$\alpha\,\alpha\,\alpha\,\alpha$
$\beta\beta\beta$	$\beta\gamma$	$\beta\beta\beta$	$\beta\,.\,\gamma$
γ	γ	$\gamma\gamma$	γ
δ		δ	

quarum numeri sunt

9	6	9	7

qui complexionibus datis præfigi jubent signa

−	+	−	− ."

The proof that this rule of signs, which is manifestly nothing else than Cramer's, leads to the same results as the previous rule, is quite easily understood if a particular permutation be first considered. For example, let the sign of the particular permutation $\delta\beta\alpha\gamma$ be wanted. Following the first rule, we should require to note four different members, viz.,

(1)	the no. of the	column in which	$\delta\beta\alpha\gamma$	occurs in the	4th	group,	
(2)	"	"	$\delta\beta\alpha$	"	3rd	"	
(3)	"	"	$\delta\beta$	"	2nd	"	
(4)	"	"	δ	"	1st	"	.

The first of these numbers being 1, we should infer that in fixing the sign of $\delta\beta\alpha\gamma$ in the fourth group there had been no change from the sign of $\delta\beta\alpha$ in the third group; the second number being also 1, we should make a like inference; the third number being 2, we should infer that in fixing the sign of $\delta\beta$ in the second group there had been 1 change from the sign of δ in the first group; and finally, the fourth number being 4, we should infer that in fixing the sign of δ in the first group there had been 3 changes from the sign of α in that group. The total number of changes from the sign of α in the first group being thus $3+1+0+0$, *i.e.*, 4, the sign would be made $+$. Now the 3 in this aggregate is simply the number of letters in the first group which precede δ, the 1 is simply the number of letters taken along with δ before β comes to be taken along with it to form $\delta\beta$ in the second group, and the two zeros correspond to the fact that $\delta\beta\alpha$ on the third group and $\delta\beta\alpha\gamma$ on the fourth group have no permutation standing to the left of them. Consequently to count the number of changes $(3+1+0+0)$ from the

sign of a in accordance with the first rule is the same as to count the number of letters placed under the given permutation, thus,

$$\begin{array}{l}\underline{\delta\beta a\gamma}\\ a\,a\,..\\ \beta\\ \gamma\end{array}$$

in accordance with the second rule.

Another point of resemblance between Rothe and De Prasse is thus made manifest, viz., that they both refused to accept Cramer's rule of signs as fundamental, preferring to base their work on a rule equally arbitrary, and then to deduce Cramer's from it.

In case it may have escaped the reader, attention may likewise be drawn to the fact that De Prasse prefixes a sign not only to permutations involving all the letters dealt with, but also to any permutation whatever involving a less number; so that in reckoning the sign of $a\delta\beta$, say, the full number of letters from which a, δ, β are chosen must be known.

A theorem like Hindenburg's is next given, viz., *If the permutations of any group be separated into sub-groups,* (1) *those which begin with* a, (2) *those which begin with* β, *and so on, then the series of signs of the* 3*rd,* 5*th, and other odd sub-groups is identical with the series of signs of the* 1*st sub-group, and the signs of any one of the even sub-groups is got by changing each sign of the first sub-group into the opposite sign.* (III. 16.)

It is more extensive than Hindenburg's in that it is true of permutations which involve less than all the letters, provided such permutations have had their signs fixed in accordance with De Prasse's rule. The proof depends, of course, on the first rule of signs, and consists in showing that if the theorem be true for any group it must, by the said rule, be true for the next group. It will be remembered that Hindenburg gave no proof.

Following this is Rothe's theorem regarding the interchange of two elements of a permutation, or rather an extension of the theorem to signed permutations involving less than the whole number of letters. The proof is as lengthy as Rothe's, even more unnecessary letters than Rothe's c, f, e being introduced. (III. 17.)

The last theorem is Vandermonde's (XII.); and this is followed by

two pages of application to the solution of simultaneous linear equations.

No reference is made by De Prasse to Hindenburg, Rothe, or Vandermonde.

WRONSKI (1812).

[Réfutation de la Théorie des Fonctions Analytiques de Lagrange. Par Höené Wronski, pp. 14, 15, . . . , 132, 133. Paris.]

In 1810 Wronski presented to the Institute of France a memoir on the so-called *Technie de l'Algorithmie,* which with his usual sanguine enthusiasm he viewed as the essential part of a new branch of Mathematics. It contained a very general theorem, now known as "Wronski's theorem," for the expansion of functions,—a theorem requiring for its expression the use of a notation for what Wronski styled *combinatory sums.* The memoir consisted merely of a statement of results, and probably on this account, although favourably reported on by Lagrange and Lacroix, was not printed. The subject of it, however, turns up repeatedly in the *Réfutation* printed two years later; and from the indications there given we can so far form an idea of the grasp which Wronski had of the theory of the said *sums.*

At page 14 the following passage occurs:—

"Soient X_1, X_2, X_3, &c. plusieurs fonctions d'une quantité variable. Nommons *somme combinatoire*, et désignons par la lettre hébraïque *sin*, de la manière que voici

$$\text{ש}[\Delta^a X_1 . \Delta^b X_2 . \Delta^c X_3 . \;.\;.\; \Delta^p X_\pi], \quad (\text{xv. } 3)\ (\text{vii. } 4)$$

la somme des produits des différences de ces fonctions, composés de la manière suivante: Formez, avec les exposans $a, b, c, \ldots, p$ des différences dont il est question, toutes les permutations possibles; donnez ces exposans, dans chaque ordre de leurs permutations, aux différences consécutives qui composent le produit

$$\Delta X_1 . \Delta X_2 . \Delta_3 . \;.\;.\; \Delta X_\pi ;$$

donnez de plus, aux produits séparés, formés de cette manière, le signe positif lorsque le nombre de variations des exposans [illegible]és dans leur ordre alphabétique, est nul ou

pair, et le signe négatif lorsque ce nombre de variations est impair; enfin, prenez la somme de tous ces produits séparés.— Vous aurez ainsi, par exemple,

$$\text{ש}[\Delta^a X_1] = \Delta^a X_1 ,$$

$$\text{ש}[\Delta^a{}_1 \, . \, \Delta^b X_2] = \Delta^a X_1 . \Delta^b X_2 - \Delta^b X_1 \, . \, \Delta^a X_2 ,$$

. . . ."

The new name, *combinatory sum*, and the new notation, did not originate in ignorance of the work of previous investigators, for memoirs of Vandermonde and Laplace are referred to. The only fresh and real point of interest lies in the fact that the first index of every pair of indices is not attached to the same letter as the second index, but belongs to an operational symbol preceding this letter, and is used for the purpose of denoting repetition of the operation. This and the allied fact that the elements are not all independent of each other, $\Delta^1 X_1$ and $\Delta^2 X_1$, for example, being connected by the equation

$$\Delta^2 X_1 = \Delta(\Delta^1 X_1),$$

indicate that Wronski's combinatory sums form a special class with properties peculiar to themselves.

BINET (November 1812).

[Mémoire sur un système de formules analytiques, et leur application à des considérations géométriques. *Journ. de l'Éc. Polyt.*, ix. cah. 16, pp. 280–302, . . .]

It would seem as if the above-noted frequent recurrence of functions of the same kind had led Binet to a special study of them. In the memoir we have now come to, his standpoint towards them is changed. They are viewed as functions having a history: for information regarding them, the writings of Vandermonde, Laplace, Lagrange, and Gauss are referred to: they are spoken of by Laplace's name for them, *résultantes à deux lettres, à trois lettres, à quatre lettres*, &c.; and the first twenty-three pages of the memoir are devoted expressly to establishing new theorems regarding them.

Of these the fundamental, and by far the most notable, is the afterwards well-known *multiplication-theorem*. It is enunciated at the outset as follows:—

"Lorsqu'on a deux systèmes de n lettres chacun, et nous supposerons chaque système écrit avec une seule lettre portant divers accens, qui serviront à ranger dans le même ordre les deux systèmes; on peut former avec ces lettres un nombre $n\frac{n-1}{2}$ de résultantes à deux lettres, en ne prenant dans le second terme de chacune que des lettres portant les mêmes accens que celles du premier. Si, avec deux autres systèmes de lettres, on forme encore des résultantes à deux lettres, et qu'on les multiplie chacune par sa correspondante obtenue des deux premiers systèmes, c'est-à-dire, par celle dont les lettres portent les mêmes accens; la somme des produits de toutes ces résultantes correspondantes sera elle-même une résultante à deux lettres, dont les termes ou lettres seront des sommes de produits des élémens des deux systèmes portant les mêmes accens. Avec deux groupes de trois systèmes de n lettres chacun, on peut former semblablement deux séries de résultantes à trois lettres; faisant ensuite la somme des produits de celles qui se correspondent par les accens de leurs lettres, on aura encore une résultante à trois lettres. Pareille chose ayant lieu pour des résultantes à quatre lettres, &c., on peut conclure ce théorème: Le produit d'un nombre quelconque de sommes de produits* de deux résultantes correspondantes de même ordre, est encore une résultante de cet ordre."

(XVII. 4 + XVIII. 4.)

The mode of proof adopted is lengthy, laborious, and not very satisfactory, except as affording a verification of the theorem for the cases of "résultantes" of low orders. It rests too on certain identities, the demonstration of which is open to similar criticism. All that Binet says regarding these absolutely essential identities is (p. 284)—

"Je représenterai par Σa la somme $a' + a'' + a''' + \&c.$, des quantités a', a'', a''', &c.; par Σab la somme des produits $ab + a'b' + a''b'' + \&c.$, dans chacun desquels les lettres a et b ont le même accent; par $\Sigma ab'$ la somme $a'b'' + b'a'' + a'b''' + \&c.$,

* There is an extension here which one is scarcely prepared for, viz., "*le produit d'un nombre quelconque de sommes de produits,*" instead of *la somme d'un nombre de produits.*

là tous les produits d'un des a par un des b, portent un accent différent de celui de a; par $\Sigma ab'c''$ la somme $a'b''c''' + b'c''a''' + c'a''b''' +$ &c., et ainsi de suite. Cela posé, on vérifie aisément les formules suivantes:

$$\Sigma ab' = \Sigma a \Sigma b - \Sigma ab\,,$$

$$\Sigma ab'c'' = \Sigma a \Sigma b \Sigma c + 2\Sigma abc - \Sigma a \Sigma bc - \Sigma b \Sigma ca - \Sigma c \Sigma ab\,,$$

$$\begin{aligned} \Sigma ab'c''d''' = {} & \Sigma a \Sigma b \Sigma c \Sigma d \quad - 6\Sigma abcd \\ & - \Sigma a \Sigma b \Sigma cd - \Sigma a \Sigma c \Sigma bd - \Sigma a \Sigma d \Sigma bc \\ & - \Sigma c \Sigma d \Sigma ab - \Sigma b \Sigma d \Sigma ac - \Sigma b \Sigma c \Sigma ad \\ & + \Sigma ab \Sigma cd + \Sigma ac \Sigma bd + \Sigma ab^* \Sigma bc \\ & + 2\Sigma a \Sigma bcd + 2\Sigma b \Sigma cda + 2\Sigma c \Sigma dab \\ & + 2\Sigma d \Sigma abc\,, \end{aligned}$$

$$\Sigma ab'c''d'''e^{\text{iv}} = \Sigma a \Sigma b \Sigma c \Sigma d \Sigma e + \text{\&c.},$$

&c.

It is thus seen that not only is no general proof of the identities given, but that even the law of formation of the right-hand members of the identities themselves is left undivulged. The exact words employed in the demonstration of the first case of the multiplication-theorem are (p. 286)—

"Avec un nombre n de lettres y', y'', y''', &c. et un même nombre de z', z'', z'', &c. on peut former $n\frac{n-1}{2}$ résultantes à deux lettres (y', z''), (y', z'''), &c. (y'', z''') &c.; ayant formé pareillement avec les lettres, v', v'', v''', &c., $\zeta', \zeta'', \zeta'''$ &c., les résultantes (v', ζ''), (v', ζ'''), &c., (v'', ζ'''), &c., considérons la somme $\Sigma(y, z')(v, \zeta')$ des produits des résultantes qui se correspondent par les accens dans les deux systèmes. On voit, en développant, par la multiplication, chacun des termes de cette somme, qu'elle revient à

$$\Sigma yv.z'\zeta' - \Sigma zv.y'\zeta'\,.$$

A ces deux dernières intégrales, on peut appliquer la transformation indiquée par la première des formules de l'art. 1: on parvient ainsi à

$$\Sigma(y, z')(v, \zeta') = \Sigma yv \Sigma z\zeta - \Sigma zv \Sigma y\zeta\,.$$

Ce dernier membre pouvant être assimilé à la forme (y, z'), il

* Meant for Σad.

en résulte que le produit d'un nombre quelconque de fonctions, telles que $\Sigma(y, z')(v, \zeta')$, est lui-même de la forme (y, z')."

The application here of the identity

$$\Sigma ab' = \Sigma a \Sigma b - \Sigma ab$$

requires a little attention. The result of multiplication and classification of the terms is

$$\Sigma yv.z'\zeta' \ - \ \Sigma zv.y'\zeta' ,$$

or, as it might preferably be written,

$$\Sigma\{\overline{yv}\,.\,\overline{z\zeta'}\} - \{\Sigma\overline{zv}\,.\,\overline{y\zeta'}\} ;$$

and this we know from the said identity

$$= [\Sigma\overline{yv}\,.\,\Sigma\overline{z\zeta} - \Sigma(\overline{yv}\,.\,\overline{z\zeta})] - [\Sigma\overline{zv}\,.\,\Sigma\overline{y\zeta} - \Sigma(\overline{zv}\,.\,\overline{y\zeta})] ,$$

which, because of the equality of $\Sigma(\overline{yv}\,.\,\overline{z\zeta})$ and $\Sigma(\overline{zv}\,.\,\overline{y\zeta})$, becomes

$$\Sigma\overline{yv}\,.\,\Sigma\overline{z\zeta} \ - \ \Sigma\overline{zv}\,.\,\Sigma\overline{y\zeta} .$$

The inherent weak points, however, of the mode of demonstration stand out more clearly when the next case comes to be considered, viz., the case for resultants of the third order. From the three sets of n letters

$$\begin{array}{llll} x, & x', & x'', & \ldots\ldots \\ y, & y', & y'', & \ldots\ldots \\ z, & z', & z'', & \ldots\ldots \end{array}$$

all possible "résultantes à trois lettres" are formed, and each resultant is multiplied by the corresponding resultant formed from other three sets of n letters,

$$\begin{array}{llll} \xi, & \xi', & \xi'', & \ldots\ldots\ldots \\ v, & v', & v'', & \ldots\ldots\ldots \\ \zeta, & \zeta', & \zeta'', & \ldots\ldots\ldots \end{array}$$

Each of these $\frac{1}{6}n(n-1)(n-2)$ products consists of 36 terms, there being thus $6n(n-1)(n-2)$ terms in all. But these $6n(n-1)(n-2)$ terms are found to be separable into six groups, viz.

$$+\Sigma\{x\xi\,.\,y'v'\,.\,z''\zeta''\} , \ +\Sigma\{y\xi\,.\,z'v'\,.\,x''\zeta''\} , \ \ldots\ldots$$

so that the result which we are able to register at this point is

$$\begin{aligned} \Sigma(x,y',z'')(\xi,v',\zeta'') = \ & \Sigma x\xi\,.\,y'v'\,.\,z''\zeta'' + \Sigma y\xi\,.\,z'v'\,.\,x''\zeta'' \\ & + \Sigma z\xi\,.\,x'v'\,.\,y''\zeta'' - \Sigma x\xi\,.\,z'v'\,.\,y''\zeta'' \\ & - \Sigma y\xi\,.\,x'v'\,.\,z''\zeta'' - \Sigma z\xi\,.\,y'v'\,.\,x''\zeta'' . \end{aligned}$$

To the right hand member of this the substitution

$$\Sigma ab'c'' = \Sigma a \Sigma b \Sigma c + 2\Sigma abc - \Sigma a \Sigma bc - \Sigma b \Sigma ca - \Sigma c \Sigma ab$$

is now applied six times in succession; that is to say, for

$$\Sigma x\xi \,.\, y'v' \,.\, z''\zeta''$$

and the five other term-aggregates which follow, we substitute

$$\Sigma x\xi \Sigma yv \Sigma z\zeta + 2\Sigma(x\xi \,.\, yv \,.\, z\zeta)$$
$$- \Sigma x\xi \Sigma(yv \,.\, z\zeta) - \Sigma yv \Sigma(z\zeta \,.\, x\xi) - \Sigma z\zeta \Sigma(x\xi \,.\, yv)$$

and five other like expressions. By this means we arrive, "toute réduction faite," at

$$\begin{aligned}\Sigma(x,y',z'')(\xi,v',\zeta'') = {} & \Sigma x\xi \Sigma yv \Sigma z\zeta + \Sigma y\xi \Sigma zv \Sigma x\zeta + \Sigma z\xi \Sigma xv \Sigma y\zeta \\ & - \Sigma x\xi \Sigma zv \Sigma y\zeta - \Sigma y\xi \Sigma xv \Sigma z\zeta - \Sigma z\xi \Sigma yv \Sigma x\zeta,\end{aligned}$$

which is the result desired.

It is easy to imagine the troubles in store for any one who might have the hardihood to attempt to establish the next case in the same manner.

If Binet's multiplication-theorem be described as expressing *a sum of products of resultants as a single resultant*, his next theorem may be said to give *a sum of products of sums of resultants as a sum of resultants.* The paragraph in regard to it is a little too much condensed to be perfectly clear, and must therefore be given verbatim. It is (p. 288)—

"Désignons par $S(y',z'')$ une somme de résultantes, telle que

$$(y_{,}',z_{,}'') + (y_{,,}',z_{,,}'') + (y_{,,,}',z_{,,,}'') + \&c.\,;$$

c'est-à-dire,

$$y_{,}'z_{,}'' - z_{,}'y_{,}'' + y_{,,}'z_{,,}'' - z_{,,}'y_{,,}'' + y_{,,,}'z_{,,,}'' - z_{,,,}'y_{,,,}'' + \&c.\,;$$

et continuons d'employer la caractéristique Σ pour les intégrales relatives aux accens supérieurs des lettres. L'expression $\Sigma[S(y,z') \,.\, S(v,\zeta')]$ devient par le développement de chacun de ses termes, et en vertu de la première formule de l'art. 1 ou de celle du no. 4,

$$\begin{aligned}& \Sigma y_{,}v_{,}\Sigma z_{,}\zeta_{,} - \Sigma z_{,}v_{,}\Sigma y_{,}\zeta_{,} + \Sigma y_{,,}v_{,}\Sigma z_{,,}\zeta_{,} - \Sigma z_{,,}v_{,}\Sigma y_{,,}\zeta_{,} + \&c. \\ + {} & \Sigma y_{,}v_{,,}\Sigma z_{,}\xi_{,,} - \Sigma z_{,}v_{,,}\Sigma y_{,}\zeta_{,,} + \Sigma y_{,,}v_{,,}\Sigma z_{,,}\zeta_{,,} - \Sigma z_{,,}v_{,,}\Sigma y_{,,}\zeta_{,,} + \&c. \\ + {} & \&c.\end{aligned}$$

En indiquant donc par S_1 des intégrales qui supposent, dans chaque terme, les mêmes accens inférieurs aux lettres du même alphabet, ces accens pouvant être ou non les mêmes pour celles des alphabets différens, on pourra écrire la précédente suite, en faisant usage de ce signe, ce qui donne

$$\Sigma[S(y, z')S(v,\zeta')] = S_1[\Sigma yv\Sigma z\zeta - \Sigma zv\Sigma y\zeta].$$

Cette nouvelle quantité est encore de la forme $S(y', z'')$, en sorte qu'on peut dire que le produit de fonctions, telles que

$$\Sigma\{S(y, z')\,S(v, \zeta')\},$$

sera lui-même de la forme $S(y', z'')$."

This, if I understand it correctly, may be paraphrased and expanded as follows:—

Take the product of two sums of s resultants, viz.

$$\{|y_1{}^1z_1{}^2| + |y_2{}^1z_2{}^2| + |y_3{}^1z_3{}^2| + \ldots\ldots + |y_s{}^1z_s{}^2|\}$$
$$\times\{|v_1{}^1\zeta_1{}^2| + |v_2{}^1\zeta_2{}^2| + |v_3{}^1\zeta_3{}^2| + \ldots\ldots + |v_s{}^1\zeta_s{}^2|\}$$

or
$$\sum_{s=1}^{s=s} |y_s{}^1z_s{}^2| \,.\, \sum_{s=1}^{s=s} |v_s{}^1\zeta_s{}^2|,$$

where, it will be observed, all the resultants in the first factor are obtained from the first resultant $|y_1{}^1z_1{}^2|$ by merely changing the lower indices into $2, 3, \ldots, s$ in succession, and that the second factor is got from the first by writing v for y and ζ for z. Then form all the like products whose first factors are

$$|y_1{}^1z_1{}^3|,\ |y_1{}^1z_1{}^4|,\ \ldots,\ |y_1{}^{n-1}z_1{}^n|;$$

these being along with $|y_1{}^1z_1{}^2|$ the $\frac{1}{2}n(n-1)$ resultants derivable from the two sets of n quantities

$$y_1{}^1,\ y_1{}^2,\ y_1{}^3,\ \ldots\ldots,\ y_1{}^n$$
$$z_1{}^1,\ z_1{}^2,\ z_1{}^3,\ \ldots\ldots,\ z_1{}^n.$$

The sum of these $\frac{1}{2}n(n-1)$ products may be represented, if we choose, by

$$\sum_{\substack{n=2\\ m<n}}^{n=n}\left[\sum_{s=1}^{s=s}|y_s{}^mz_s{}^n| \,.\, \sum_{s=1}^{s=s}|v_s{}^m\zeta_s{}^n|\right].$$

Now if the multiplications be performed, there will be s^2 terms in each product, and the theorem we are concerned with has its origin in the fact that the sum of all the first terms of the products is

expressible as a resultant by applying the multiplication-theorem, likewise the sum of all the second terms, and so on, the result being an aggregate of s^2 resultants. For if we fix upon a particular term of the first product, say the term

$$|y_h^1 z_h^2| . |v_k^1 \zeta_k^2|$$

which arises from the multiplication of the h^{th} term of the first factor by the k^{th} term of the second factor, then take the corresponding term of the other products, and write down their sum

$$|y_h^1 z_h^2| . |v_k^1 \zeta_k^2| + |y_h^1 z_h^3| . |v_k^1 \zeta_k^3| + \ldots\ldots + |y_h^{n-1} \zeta_h^n| . |v_k^{n-1} \zeta_k^n|,$$

it is manifest that this sum is by the multiplication-theorem

$$= \begin{vmatrix} y_h^1 v_k^1 + y_h^2 v_k^2 + \ldots\ldots + y_h^n v_k^n & z_h^1 v_k^1 + z_h^2 v_k^2 + \ldots\ldots + z_h^n v_k^n \\ y_h^1 \zeta_k^1 + y_h^2 \zeta_k^2 + \ldots\ldots + y_h^n \zeta_k^n & z_h^1 \zeta_k^1 + z_h^2 \zeta_k^2 + \ldots\ldots + z_h^n \zeta_k^n \end{vmatrix}.$$

Consequently since h may be any integer from 1 to s, and k likewise any integer from 1 to s, the theorem arrived at is accurately expressed in modern notation as follows:—

$$\sum_{\substack{n=2 \\ m < n}}^{n=n} \left[\sum_{s=1}^{s=s} |y_s^m z_s^n| . \sum_{s=1}^{s=s} |v_s^m \zeta_s^n| \right]$$

$$= \sum_{k=1}^{k=s} \sum_{h=1}^{h=s} \begin{vmatrix} y_h^1 v_k^1 + y_h^2 v_k^2 + \ldots + y_h^n v_k^n & z_h^1 v_k^1 + z_h^2 v_k^2 + \ldots + z_h^n v_k^n \\ y_h^1 \zeta_k^1 + y_h^2 \zeta_k^2 + \ldots + y_h^n \zeta_k^n & z_h^1 \zeta_k^1 + z_h^2 \zeta_k^2 + \ldots + z_h^n \zeta_k^n \end{vmatrix},$$

or

$$\sum_{k=1}^{k=s} \sum_{h=1}^{h=s} \begin{vmatrix} y_h^1 & y_h^2 & \ldots\ldots & y_h^n \\ z_h^1 & z_h^2 & \ldots\ldots & z_h^n \end{vmatrix} . \begin{vmatrix} v_k^1 & v_k^2 & \ldots\ldots & v_k^n \\ \zeta_k^1 & \zeta_k^2 & \ldots\ldots & \zeta_k^n \end{vmatrix}.$$

It is easily seen to be true of resultants of any order, as Binet himself points out. (xxx.)

When s is put equal to 1, it degenerates into the multiplication-theorem.

The theorem which follows upon this, but which is quite unconnected with it, may be at once stated in modern notation. It is—

If $\Sigma|x_1 y_2 z_3|$ denote the sum of the resultants obtainable from the three sets of n quantities

$$\begin{matrix} x_1 & x_2 & x_3 & \ldots\ldots & x_n \\ y_1 & y_2 & y_3 & \ldots\ldots & y_n \\ z_1 & z_2 & z_3 & \ldots\ldots & z_n, \end{matrix}$$

and $\Sigma|x_1y_2|$ denote the like sum obtainable from the first two sets, then

$$\Sigma|x_1y_2z_3| = \Sigma x.\Sigma|y_1z_2| + \Sigma y.\Sigma|z_1x_2| + \Sigma z.\Sigma|x_1y_2| \quad \text{(XXXI.)}$$

This is arrived at by writing out the terms of $\Sigma|y_1z_2|$, of $\Sigma|z_1x_2|$, and of $\Sigma|x_1y_2|$ in parallel columns, thus

$$\begin{array}{ccc} |y_1\ z_2| & |z_1\ x_2| & |x_1\ y_2| \\ |y_1\ z_3| & |z_1\ x_3| & |x_1\ y_3| \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ |y_{n-1}z_n| & |z_{n-1}x_n| & |x_{n-1}y_n|\,; \end{array}$$

then deriving n results from the members of the first row by multiplying by x_1, y_1, z_1 respectively and adding, multiplying by x_2, y_2, z_2, and adding, and so on; then treating the second and remaining rows in the same way; and then finally adding all the $n\,.\,\frac{1}{2}n(n-1)$ results together. Each of these results is a vanishing or non-vanishing resultant of the 3rd order, and it will be found that each non-vanishing resultant occurs twice with the sign $+$ and once with the sign $-$.

This process is readily seen to be simply the same as performing the multiplications indicated in the right-hand member of (XXXI.), *i.e.*,

$$\begin{aligned} &(x_1+x_2+\ldots+x_n)\,(|y_1z_2| + |y_1z_3| + \ldots + |y_{n-1}z_n|) \\ +&(y_1+y_2+\ldots+y_n)\,(|z_1x_2| + |z_1x_3| + \ldots + |z_{n-1}x_n|) \\ +&(z_1+z_2+\ldots+z_n)\,(|x_1y_2| + |x_1y_3| + \ldots + |x_{n-1}y_n|)\,, \end{aligned}$$

summing every three corresponding terms in the products, and writing the sum as a vanishing or non-vanishing resultant. There would be $n\,.\,\frac{1}{2}n(n-1)$ resultants in all; but as each suffix occurs $n-1$ times in the second factors and once in the first factors, there must be in each product $n-1$ terms having the said suffix occurring twice: consequently there must be $n-1$ resultants vanishing on account of this recurrence, and therefore altogether $n(n-1)$ vanishing resultants. Of the non-vanishing resultants,—in number equal to $n\,.\,\frac{1}{2}n(n-1)-n(n-1)$, or $\frac{1}{2}n(n-1)(n-2)$,—each one of the form

$$|x_hy_kz_l| \qquad \text{where } h<k<l$$

must be accompanied by two others,

$$| x_k y_h z_l | \text{ and } | x_l y_h z_k |,$$

and the sum of these is

$$| x_h y_k z_l | - | x_h y_k z_l | + | x_h y_k z_l |,$$

i.e.,

$$| x_h y_k z_l |.$$

The final result is thus the sum of the resultants of the form

$$| x_h y_k z_l | \text{ where } h < k < l, \text{ and } l = 3, 4, \ldots, n,$$

the number of them, as we may see from two different standpoints, being

$$\tfrac{1}{6}n(n-1)(n-2).$$

Returning to the series of identities,

$$\begin{aligned} x_3|y_1z_2| + y_3|z_1x_2| + z_3|x_1y_2| &= |x_1y_2z_3|, \\ x_4|y_1z_2| + y_4|z_1x_2| + z_4|x_1y_2| &= |x_1y_2z_4|, \\ \&c. \qquad\qquad &\&c. \end{aligned}$$

which by addition give the result

$$\Sigma x\Sigma|y_1z_2| + \Sigma y\Sigma|z_1x_2 + \Sigma z\Sigma|x_1y_2| = \Sigma|x_1y_2z_3|,$$

Binet next raises both sides of all of them to the second power, and obtains

$$\left.\begin{aligned} 3\Sigma|x_1y_2z_3|^2 = \Sigma x^2\Sigma|y_1z_2|^2 + \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma z^2\Sigma|x_1y_2|^2 \\ + 2\Sigma yz\Sigma(|z_1x_2|.|x_1y_2|) + 2\Sigma zx\Sigma(|x_1y_2|.|y_1z_2|) \\ + 2\Sigma xy\Sigma(|y_1z_2|.|z_1x_2|). \end{aligned}\right\} \text{(XXXII.)}$$

Substituting for $\Sigma|y_1z_2|^2$, $\Sigma|z_1x_2|^2$, ... , their equivalents as given by the multiplication-theorem, he then deduces

$$\left.\begin{aligned} \Sigma|x_1y_2z_3|^2 = \Sigma x^2\Sigma y^2\Sigma z^2 + 2\Sigma yz\Sigma zx\Sigma xy - \Sigma x^2(\Sigma z^x)^2 \\ - \Sigma y^2(\Sigma zx)^2 - \Sigma z^2(\Sigma xy)^2, \end{aligned}\right\}$$

not failing to note that this is not a fresh result, but merely a case of the multiplication-theorem in which the factors are equal.

By putting the right-hand member here into the form

$$\begin{aligned} \Sigma y^2\{\Sigma z^2\Sigma x^2 - (\Sigma yz)^2\} + \Sigma z^2\{\Sigma x^2\Sigma y^2 - (\Sigma xy)^2\} \\ - \Sigma x^2\{\Sigma y^2\Sigma z^2 - (\Sigma yz)^2\} + 2\Sigma yz\{\Sigma zx\Sigma xy - \Sigma yz\Sigma x^2\}, \end{aligned}$$

there is next arrived at the first identity of the set

$$\left.\begin{aligned}\Sigma|x_1y_2z_3|^2 \\ = \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma z^2\Sigma|x_1y_2|^2 - \Sigma x^2\Sigma|y_1z_2|^2 + 2\Sigma yz\Sigma|z_1x_2||x_1y_2|, \\ = \Sigma z^2\Sigma|x_1y_2|^2 + \Sigma x^2\Sigma|y_1z_2|^2 - \Sigma y^2\Sigma|z_1x_2|^2 + 2\Sigma zx\Sigma|x_1y_2||y_1z_2|, \\ = \Sigma x^2\Sigma|y_1z_2|^2 + \Sigma y^2\Sigma|z_1x_2|^2 - \Sigma z^2\Sigma|x_1y_2|^2 + 2\Sigma xy\Sigma|y_1z_2||z_1x_2|,\end{aligned}\right\}\text{(XXXIII.)}$$

and immediately from these the set

$$\left.\begin{aligned}\Sigma|x_1y_2z_3|^2 &= \Sigma x^2\Sigma|y_1z_2|^2 + \Sigma zx\Sigma|x_1y_2|\cdot|y_1z_2| + \Sigma xy\Sigma|y_1z_2|\cdot|z_1x_2|, \\ &= \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma xy\Sigma|y_1z_2|\cdot|z_1x_2| + \Sigma yz\Sigma|z_1x_2|\cdot|x_1y_2|, \\ &= \Sigma z^2\Sigma|x_1y_2|^2 + \Sigma yz\Sigma|z_1x_2|\cdot|x_1y_2| + \Sigma zx\Sigma|x_1y_2|\cdot|y_1z_2|.\end{aligned}\right\}\text{(XXXIV.)}$$

We may note in passing that either of these sets leads at once to the initial theorem

$$\begin{aligned}3\Sigma|x_1y_2z_3|^2 = \Sigma x^2\Sigma|y_1z_2|^2 + \Sigma y^2\Sigma|z_1x_2|^2 + \Sigma z^2\Sigma|x_1y_2|^2 \\ + 2\Sigma yz\Sigma|z_1x_2|\cdot|x_1y_2| + 2\Sigma zx\Sigma|x_1y_2|\cdot|y_1z_2| \\ + 2\Sigma xy\Sigma|y_1z_2|\cdot|z_1x_2|,\end{aligned}$$

and that with the multiplication-theorem already established this reverse order would be the more natural.

The next step taken is the formation of resultants of the 2[nd] order from elements which are themselves resultants of the 2[nd] order; viz., just as from the three rows of n quantities

$$\begin{matrix} x_1 & x_2 & x_3 & \dots & x_n \\ y_1 & y_2 & y_3 & \dots & y_n \\ z_1 & z_2 & z_3 & \dots & z_n \end{matrix}$$

there were formed the three other rows of $\frac{1}{2}n(n-1)$ quantities

$$\begin{matrix} |y_1z_2|, & |y_1z_3|, & \dots, & |y_1z_n|, & |y_2z_3|, & \dots, & |y_{n-1}z_n|, \\ |z_1x_2|, & |z_1x_3|, & \dots, & |z_1x_n|, & |z_2x_3|, & \dots, & |z_{n-1}x_n|, \\ |x_1y_2|, & |x_1y_3|, & \dots, & |x_1y_n|, & |x_2y_3|, & \dots, & |x_{n-1}y_n|, \end{matrix}$$

so from the latter three other rows of quantities

$$\begin{Vmatrix} |z_1x_2| & |z_1x_3| \\ |x_1y_2| & |x_1y_3| \end{Vmatrix}, \dots\dots, \begin{Vmatrix} |z_{n-2}x_n| & |z_{n-1}x_n| \\ |x_{n-2}y_n| & |x_{n-1}y_n| \end{Vmatrix},$$

$$\begin{Vmatrix} |x_1y_2| & |x_1y_3| \\ |y_1z_2| & |y_1z_3| \end{Vmatrix}, \dots\dots, \begin{Vmatrix} |x_{n-2}y_n| & |x_{n-1}y_n| \\ |y_{n-2}z_n| & |y_{n-1}z_n| \end{Vmatrix},$$

$$\begin{Vmatrix} |y_1z_2| & |y_1z_3| \\ |z_1x_2| & |z_1x_3| \end{Vmatrix}, \dots\dots, \begin{Vmatrix} |y_{n-2}z_n| & |y_{n-1}z_n| \\ |z_{n-2}x_n| & |z_{n-1}x_n| \end{Vmatrix},$$

are formed, the number in each new row being clearly

$$\tfrac{1}{2}\{\tfrac{1}{2}n(n-1)\}\{\tfrac{1}{2}n(n-1)-1\}$$
$$i.e.,\qquad \tfrac{1}{8}n(n-1)(n-2)(n+1).$$

The new quantities are, of course, not written by Binet in the form

$$\left|\begin{matrix} | & | & | & | \\ | & | & | & | \end{matrix}\right|,$$

but the fact that they are resultants of the 2nd order is carefully noted. Each of them is shown to be transformable, by a theorem which may be viewed as an extension of a result given by Lagrange, so as to have two of the elements resultants of the 3rd order, and the others resultants of the 1st order. This is done by taking, for example, the identities

$$x_h|y_iz_j| + y_h|z_ix_j| + z_h|x_iy_j| = |x_hy_iz_j|,$$
$$x_k|y_iz_j| + y_k|z_ix_j| + z_k|x_iy_j| = |x_ky_iz_j|,$$

multiplying both sides of the first by x_k, and both sides of the second by x_h, subtracting, and writing the result in the form

$$|x_ky_h||z_ix_j| + |x_kz_h||x_iy_j| = x_k|x_hy_iz_j| - x_h|x_ky_iz_j|,$$
$$= \begin{vmatrix} x_k & x_h \\ |x_ky_iz_j| & |x_hy_iz_j| \end{vmatrix},$$

where of course it has to be noted that in many cases one of the resultants of the 3rd order will vanish. The quantities, therefore, to be dealt with, are

$$x_1|x_1y_2z_3|,\ \ldots\ldots,\ x_k|x_hy_iz_j| - x_h|x_ky_iz_j|,\ \ldots\ldots,\ x_n|x_{n-2}y_{n-1}z_n|;$$
$$y_1|x_1y_2z_3|,\ \ldots\ldots,\ y_k|y_hz_ix_j| - y_h|y_kz_ix_j|,\ \ldots\ldots,\ y_n|x_{n-2}y_{n-1}z_n|;$$
$$z_1|x_1y_2z_3|,\ \ldots\ldots,\ z_k|z_hx_iy_j| - z_h|z_kx_iy_j|,\ \ldots\ldots,\ z_n|x_{n-2}y_{n-1}z_n|.$$

By raising each of the elements of the first row to the second power, taking the sum and simplifying, we could, we are told, show that the result would be

$$\Sigma x_1^2\,\Sigma|x_1y_2z_3|^2.$$

Very prudently, however, another process is chosen. It is recalled that the quantities in the third triad of rows are related to those in the second as those in the second are related to those in the first, and that consequently the required sum of squares of resultants is, by the multiplication-theorem itself, expressible as a resultant, viz.,

$$\Sigma \Big\| |z_1x_2| \,,\ |x_1y_3| \Big\|^2 = \Sigma|z_1x_2|^2 \,.\, \Sigma|x_1y_2|^2 - \big(\Sigma|z_1x_2||x_1y_2|\big)^2 ,$$

where the elements of the resultant on the right are sums of products of quantities in the second triad of rows. Then the same theorem is used to make a further step backwards, viz., to express each of these three sums of products of resultants as a resultant whose elements are sums of products of the quantities in the first triad of rows, the effect of the substitution being

$$\begin{aligned}\Sigma \Big\| |z_1x_2| \,,\ |x_1y_3| \Big\|^2 = \{\Sigma z_1{}^2\Sigma x_1{}^2 - (\Sigma z_1x_1)^2\}\{\Sigma x_1{}^2\Sigma y_1{}^2 - (\Sigma x_1y_1)^2\} \\ - \{\Sigma z_1x_1\Sigma x_1y_1 - \Sigma y_1z_1\Sigma x_1{}^2\}^2 .\end{aligned}$$

Simple multiplication transforms this into

$$\Sigma x_1{}^2 \left\{ \begin{array}{r} \Sigma x_1{}^2\Sigma y_1{}^2\Sigma z_1{}^2 \;-\; \Sigma y_1{}^2(\Sigma z_1x_1)^2 \;-\; \Sigma z_1{}^2(\Sigma x_1y_1)^2 \\ +\,2\Sigma y_1z_1\Sigma z_1x_1\Sigma x_1y_1 \;-\; \Sigma x_1{}^2(\Sigma y_1z_1)^2 \end{array} \right\} ,$$

which, by still another use of the multiplication-theorem, we know is equal to

$$\Sigma x_1{}^2\ \Sigma|x_1y_2z_3|^2.$$

The set of six results of which this is one, is

$$\left.\begin{aligned} \Sigma X_1{}^2 &= \Sigma\, x_1{}^2\ \Sigma|x_1y_2z_3|^2 , \\ \Sigma Y_1{}^2 &= \Sigma y_1{}^2\ \Sigma|x_1y_2z_3|^2 , \\ \Sigma Z_1{}^2 &= \Sigma\, z_1{}^2\ \Sigma|x_1y_2z_3|^2 , \\ \Sigma Y_1Z_1 &= \Sigma y_1z_1\Sigma|x_1y_2z_3|^2 , \\ \Sigma Z_1X_1 &= \Sigma z_1x_1\Sigma|x_1y_2z_3|^2 , \\ \Sigma X_1Y_1 &= \Sigma x_1y_1\Sigma|x_1y_2z_3|^2 , \end{aligned}\right\} \qquad \text{(xxxv.)}$$

if, for shortness, we denote the quantities of the third triad of rows by

$$\begin{array}{lll} X_1, & X_2, & \ldots\ldots \\ Y_1, & Y_2, & \ldots\ldots \\ Z_1, & Z_2, & \ldots\ldots \end{array}$$

Following these, and deduced by means of them, is an equally noteworthy theorem regarding the sums of squares of all the resultants of the third order, which can be formed from the quantities of the second triad of rows. Denoting these quantities temporarily by

$$\begin{array}{lll} \xi_1, & \xi_2, & \ldots\ldots\ldots \\ \eta_1, & \eta_2, & \ldots\ldots\ldots \\ \zeta_1, & \zeta_2, & \ldots\ldots\ldots \end{array}$$

we know (XXXII.) that

$$3\Sigma|\xi\eta_2\zeta_3|^2 = \Sigma X_1{}^2\Sigma\xi_1{}^2 + \Sigma Y_1{}^2\Sigma\eta_1{}^2 + \Sigma Z_1{}^2\Sigma\zeta_1{}^2 + 2\Sigma Y_1Z_1\,.\,\Sigma\eta_1\zeta_1 + 2\Sigma Z_1X_1\,.\,\Sigma\zeta_1\xi_1 + 2\Sigma X_1Y_1\,.\,\Sigma\xi_1\eta_1\,;$$

whence, by using the set of six results just obtained, we have

$$3\Sigma|\xi_1\eta_2\zeta_3|^2 = \Sigma|x_1y_2z_3|^2\left\{\begin{array}{l}\Sigma\xi_1{}^2\Sigma x_1{}^2 + \Sigma\eta_1{}^2\Sigma y_1{}^2 + \Sigma\zeta_1{}^2\Sigma z_1{}^2 \\ + 2\Sigma\eta_1\zeta_1.\Sigma y_1z_1 + 2\Sigma\zeta_1\xi_1.\Sigma z_1x_1 + 2\Sigma\xi_1\eta_1.\Sigma x_1y_1\end{array}\right\}$$

and therefore, again by (XXXII.)

$$\Sigma|\xi_1\eta_2\zeta_3|^2 = \{\Sigma|x_1y_2z_3|^2\}^2. \qquad \text{(XXXVI.)}$$

It is finally pointed out that from the third triad of rows there might, in like manner, be formed a fourth triad, and analogous identities obtained; also that, instead of starting with three rows, we might start with *four*,

$$\begin{array}{l} t_1,\ t_2,\ t_3,\ \ldots\ldots,\ t_n \\ x_1,\ x_2,\ x_3,\ \ldots\ldots,\ x_n \\ y_1,\ y_2,\ y_3,\ \ldots\ldots,\ y_n \\ z_1,\ z_2,\ z_3,\ \ldots\ldots,\ z_n, \end{array}$$

form from them other four

$$\begin{array}{l} |x_1y_2z_3|,\ \ldots\ldots\ldots\ldots \\ |y_1z_2t_3|,\ \ldots\ldots\ldots\ldots \\ |z_1t_2x_3|,\ \ldots\ldots\ldots\ldots \\ |t_1x_2y_3|,\ \ldots\ldots\ldots\ldots, \end{array}$$

thence in the same way a third four, and in connection therewith establish the identity

$$\Sigma t_1\Sigma|x_1y_2z_3| - \Sigma x_1\Sigma|y_1z_2t_3| + \Sigma y_1\Sigma|z_1t_2x_3| - \Sigma z_1\Sigma|t_1x_2y_3| = 0 \quad \text{(XXXI. 2)}$$

and other analogues. (XXXII. 2 + XXXV. 2.)

The rest of the memoir, 52 pages, consists of geometrical applications of the series of theorems thus obtained.

CAUCHY (1812).

[Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions

opérées entre les variables qu'elles renferment. *Journ. de l'Ec. Polyt.*, x. Cah. 17, pp. 29–112.]

This masterly memoir of 84 pages was read to the Institute on the same day (30th November) as Binet's memoir, of which we have just given an account. The coincidence of date has to be carefully noted, because the memoirs have in part a common ground, and because there is a presumption that the authors, knowing this beforehand, had, in a friendly way, arranged for simultaneous publicity. Binet's words on the matter are (IX. p. 281)—

"Ayant eu dernièrement occasion de parler à M. Cauchy, ingénieur des ponts et chaussées, du théorème général que j'ai énoncé ci-dessus, il me dit être parvenu, dans des recherches analogues à celles de M. Gauss, à des théorèmes d'analyse qui devaient avoir rapport aux miens. Je m'en suis assuré, en jetant les yeux sur ces formules: mais j'ignore si elles ont la même généralité que les miennes: nous y sommes arrivés, je crois, par des voies très-différentes."

And Cauchy's corroboration is (p. 111)—

"J'avais rencontré l'été dernier, à Cherbourg, où j'étais fixé par les travaux de mon état, ce théorème et quelques autres du même genre, en cherchant à généraliser les formules de M. Gauss. M. Binet, dont je me félicite d'être l'ami, avait été conduit aux mêmes résultats par des recherches différentes. De retour à Paris, j'étais occupé de poursuivre mon travail, lorsque j'allai le voir. Il me montra son théorème qui était semblable au mien. Seulement il désignait sous le nom de *résultante* ce que j'avais appelé *déterminant.*"

Cauchy prefaces his memoir by another, entitled

Sur le nombre des valeurs qu'une fonction peut acquérir lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme.

This latter must to a certain extent be taken into account, because it serves to show the point of view which he considered most natural for examining the subject, and also the exact position held by the functions now called determinants, when functions in

general come to be classified according to the number of values they are able to assume in certain circumstances.

At the outset of it the writings of Lagrange, Vandermonde, and Ruffini are referred to; the fact is recalled that the maximum number of values which a function can acquire by interchanges among its n variables is $1.2.3 \ldots . n$; also that when the maximum is not obtained, the actual number must be a factor of the maximum; and then proof is given of the very notable theorem that *the number of values cannot be less than the greatest prime contained in* n *without being equal to* 2. It is pointed out likewise that functions capable of having only two values are known from Vandermonde to be constructible for any number of variables. For example, the number of variables being three, a_1, a_2, a_3, all that is needed is to form their difference-product

$$(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$$

or

$$a_3^2a_2 + a_2^2a_1 + a_1^2a_3 - (a_3^2a_1 + a_2^2a_3 + a_1^2a_2),$$

when it is found that either of the parts

$$a_3^2a_2 + a_2^2a_1 + a_1^2a_3,$$

or

$$a_3^2a_1 + a_2^2a_3 + a_1^2a_2,$$

is an instance of a function capable of only two values by permutation of the variables; the result indeed of any permutation being merely that the one function passes into the other. Further, the whole expression

$$a_3^2a_2 + a_2^2a_1 + a_1^2a_3 - (a_3^2a_1 + a_2^2a_3 + a_1^2a_2)$$

is another example, the difference between the two values which it can assume being however a difference of sign merely. As a reference to the title of the memoir of November 1812 will show, it is functions of this latter class which Cauchy there considers.

At the commencement he contrasts them with functions which suffer no change whatever by permutation of variables, that is to say, *symmetric* functions: and, noting the fact, afterwards ascertained, that the new functions consist of terms alternately + and −, and that were it not for this alternation of sign they would be symmetric functions, he decides to extend the term "symmetric" to them, and having done so, seeks to distinguish them from ordi-

nary symmetric functions by calling them "fonctions symétriques alternées," and calling the other "fonctions symétriques permanentes." Cauchy's view of determinants may therefore now be described by saying that he considered them as *a special class of alternating symmetric functions.*

To include them, however, either the adoption of a convention is necessary, or an extension of the definition must be made. For example, $a_1b_2 - a_2b_1$ is not an alternating function, unless the elements be so related that the interchange of a_1 and a_2 necessitates the interchange of b_1 and b_2 at the same time; or unless the definition be so worded that interchange shall refer to *suffixes,* not to letters. Cauchy selects the former course, his words being (p. 30)

". . . . . concevons les diverses suites de quantités

$$\begin{array}{cccc} a_1, & a_2, & \ldots\ldots, & a_n \\ b_1, & b_2, & \ldots\ldots, & b_n \\ c_1, & c_2, & \ldots\ldots, & c_n \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

tellement liées entre elles, que la transposition de deux indices pris dans l'une des suites, nécessite la même transposition dans toutes les autres; alors, les quantités

$$b_1, c_1, \ldots, b_2, c_2, \ldots, b_3, c_3, \ldots\ldots$$

pourront être considérées comme des fonctions semblables de

$$a_1, a_2, a_3, \ldots\ldots;$$

et par suite, les fonctions de

$$a_1, b_1, c_1, \ldots, a_2, b_2, c_2, \ldots, a_n, b_n, c_n, \ldots\ldots$$

qui ne changeront pas de valeur, mais tout au plus de signe, en vertu de transpositions opérées entre les indices 1, 2, 3, ... n, devront être rangées parmi les fonctions symétriques de $a_1, a_2, \ldots, a_n$, ou, ce qui revient au même, des indices 1, 2, 3, . . . , n. Ainsi

$$a_1^2 + a_2^2 + 4a_1a_2,$$

$$a_1b_1 + a_2b_2 + a_3b_3 + 2c_1c_2c_3,$$

$$a_1b_2 + a_2b_3 + a_3b_1 + a_2b_1 + a_3b_2 + a_1b_3,$$

$$\cos(a_1 - a_2)\cos(a_1 - a_3)\cos(a_2 - a_3),$$

seront des fonctions symétriques permanentes, la première du second ordre et les autres du troisième ; et au contraire,

$$a_1b_2+a_2b_3+a_3b_1-a_2b_1-a_1b_3-a_3b_2,$$
$$\sin(a_1-a_2)\sin(a_1-a_3)\sin(a_2-a_3)$$

seront des fonctions symétriques alternées du troisième ordre."

The question of nomenclature being settled there next arises the question of notation. This also is decided on the ground of the resemblance of the functions to symmetric functions. It being known that any symmetric function is representable by a typical term preceded by a symbol indicating permutation of the variables, *e.g.*

$$S(a_1b_2) \text{ or } S^2(a_1b_2) \text{ standing for } a_1b_2+a_2b_1$$

and $\quad S^3(a_1b_2)$ standing for $a_1b_2+a_2b_3+a_3b_1+a_2b_1+a_3b_2+a_1b_3$;

also, that any non-symmetric function may be taken as the typical term of a symmetric function, the question arises whether the like may not be true of alternating functions. A lengthy examination of the latter point leads to the conclusion that any non-symmetric function K *cannot* be the originating or typical term of an alternating function unless it satisfies a certain condition, viz., that it be such that any value of it obtained by an even number of transpositions of indices will be different from any other value obtained by an odd number of transpositions. Should, however, this condition be satisfied, and $K_\alpha, K_\beta, K_\gamma, \ldots.$ be all the values of the former kind, and $K_\lambda, K_\mu, K_\nu, \ldots.$ all the values of the latter kind, then

$$(K_\alpha+K_\beta+K_\gamma+\ldots.)-(K_\lambda+K_\mu+K_\nu+\ldots.)$$

is an alternating function and is appropriately representable by

$$S(\pm K)$$

if the indices appearing in K alone are to be permuted, and by

$$S^n(\pm K)$$

if the indices to be permuted be $1, 2, 3, \ldots, n$. For example, taking the typical term a_1b_2 we have

$$S(\pm a_1b_2)=a_1b_2-a_2b_1,$$

and $$S^3(\pm a_1b_2)=a_1b_2+a_2b_3+a_3b_1-a_2b_1-a_3b_2-a_1b_3,$$
$$=S^3(\mp a_2b_1) = S^3(\mp a_1b_3) = \ldots.$$

$S_4(\pm a_1 b_2)$ is an impossibility, as when there are four indices $a_1 b_2$ does not satisfy the condition required of a typical term; indeed, Cauchy notes that the number of indices in any term must either be the total number or 1 less.

The number of permutations being even, it is clear that *the number of + terms* K_α, K_β, ... *is the same as the number of negative terms* K_λ, K_μ, (x. 2)
a generalisation of a remark of Vandermonde's.

Further, since K_α, K_β, . . . are all the terms that arise from an even number of transpositions, and K_λ, K_μ, ... all those that arise from an odd number of transpositions, it is plain that any single transposition performed upon each of the terms of the function

$$(K_\alpha + K_\beta + K_\gamma + \ldots\ldots) - (K_\lambda + K_\mu + K_\nu + \ldots\ldots)$$

must change it into

$$(K_\lambda + K_\mu + K_\nu + \ldots\ldots) - (K_\alpha + K_\beta + K_\gamma + \ldots\ldots)$$

—this is, in fact, the proof that it is an alternating function—consequently each of the parts

$$K_\alpha + K_\beta + K_\gamma + \ldots\ldots,$$
$$K_\lambda + K_\mu + K_\nu + \ldots\ldots,$$

belongs to the class of functions which have only two different values.

Also it is evident that *if throughout the function any particular index be changed into another and no further alteration made, the resulting expression must be equal to zero,* (xii. 5)
a theorem regarding alternating functions which is the generalisation of a theorem of Vandermonde's.

We have lastly to note, that the criterion which determines whether a particular K belongs to the class K_α, K_β, ... or to the class K_λ, K_μ, ... is incidentally shown to be reducible to a more practical form. For example, if the term be K_θ, and it be derivable from K, say, by the change of the suffixes 1, 2, 3, 4, 5, 6, 7 into 3, 2, 6, 5, 4, 1, 7, that is to say, in Cauchy's language by means of the substitution

$$\begin{pmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7\\ 3, & 2, & 6, & 5, & 4, & 1, & 7\end{pmatrix},$$

we transform this substitution into a "product" of "circular" substitutions, viz., into

$$\begin{pmatrix}1, 3, 6\\3, 6, 1\end{pmatrix} \cdot \begin{pmatrix}4 & 5\\5 & 4\end{pmatrix} \cdot \begin{pmatrix}2\\2\end{pmatrix} \cdot \begin{pmatrix}7\\7\end{pmatrix}$$

and subtracting the number of "factors," 4, from the total number of suffixes 7, make the sign + or – according as this difference is even or odd.

Here the subject of general alternating functions may be left for the present. What remains of the first part of the memoir, refers to special cases, which naturally fall to be considered in another chapter of our history. At the close of the part, Cauchy says (p. 51)—

> "Je vais maintenant examiner particulièrement une certaine espèce de fonctions symétriques alternées qui s'offrent d'elles-mêmes dans un grand nombre de recherches analytiques. C'est au moyen de ces fonctions qu'on exprime les valeurs générales des inconnues que renferment plusieurs équations du premier degré. Elles se représentent toutes les fois qu'on a des équations à former, ainsi que dans la théorie générale de l'élimination."

The writings of Laplace, Vandermonde, Bézout, and Gauss are referred to, and from the latter the name "déterminant" is adopted.

The second part bears the title—

Des fonctions symétriques alternées désignées sous le nom de déterminans.

and opens with the following explanatory definition (p. 51)—

> "Soient $a_1, a_2, \ldots\ldots, a_n$ plusieurs quantités différentes en nombre égal à n. On a fait voir ci-dessus qu'en multipliant le produit de ces quantités, ou
>
> $$a_1 a_2 a_3 \ldots\ldots a$$
>
> par le produit de leurs différences respectives, ou par
>
> $$(a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)(a_3 - a_2)\ldots(a_n - a_2)\ldots(a_n - a_{n-1})$$
>
> on obtenait pour résultat la fonction symétrique alternée
>
> $$S(\pm a_1 a_2^2 a_3^3 \ldots\ldots a_n^n),$$

qui par conséquent se trouve toujours égale au produit

$$a_1a_2a_3 \ldots . a_n$$
$$\times (a_2-a_1)(a_3-a_1)\ldots(a_n-a_1)(a_3-a_2)\ldots(a_n-a_2)\ldots(a_n-a_{n-1}) \, .$$

Supposons maintenant que l'on développe ce dernier produit, et que dans chaque terme du développement on remplace l'exposant de chaque lettre par un second indice égal à l'exposant dont il s'agit, en écrivant par exemple $a_{r,s}$ au lieu de a_r^s, et $a_{s,r}$ au lieu de a_s^r, on obtiendra pour résultat une nouvelle fonction symétrique alternée, qui, au lieu d'être représentée par

$$S(\pm\, a_1^1a_2^2a_3^3 \;\ldots\; a_n^n)$$

sera représentée par

$$S(\pm\, a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3} \;\ldots\; a_{n\cdot n}) \, ,$$

le signe S étant relatif aux premiers indices de chaque lettre. Telle est la forme la plus générale des fonctions que je désignerai dans la suite sous le nom de *déterminans*. Si l'on suppose successivement *

$$n=1, \; n=2, \; \&c. \ldots\ldots$$

on trouvera

$$\begin{aligned} S(\pm a_{1\cdot1}a_{2\cdot2}) &= a_{1\cdot1}a_{2\cdot2} - a_{2\cdot1}a_{1\cdot2} \, , \\ S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}) &= a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} + a_{2\cdot1}a_{3\cdot2}a_{1\cdot3} + a_{3\cdot1}a_{1\cdot2}a_{2\cdot3} \\ &\quad - a_{1\cdot1}a_{3\cdot2}a_{2\cdot3} - a_{3\cdot1}a_{2\cdot2}a_{1\cdot3} - a_{2\cdot1}a_{1\cdot2}a_{3\cdot3} \, . \\ \&c. \ldots\ldots\ldots \end{aligned}$$

pour les déterminans du second, du troisième ordre, &c. . . . . "

In regard to this it is important to notice that there are really two definitions given us. The latter, viz., that involved in the symbolism of alternating functions,

$$S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} \;\ldots\; a_{n\cdot n})$$

contains nothing more than Leibnitz's rule of formation and an improved rule of signs. The former is new and may be paraphrased as follows:—

If the multiplications indicated in the expression

$$a_1a_2a_3\ldots a_n$$
$$\times (a_2-a_1)(a_3-a_1)\ldots(a_n-a_1)(a_3-a_2)\ldots(a_n-a_2)\ldots(a_n-a_{n-1})$$

* $n=2$, $n=3$, &c. is meant.

be performed, and in the result every index of a power be changed into a second suffix, e.g., a_r^s *into* $a_{r,s}$, *the expression so obtained is called a determinant* (VIII. 2),
and is denoted by

$$S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} \;.\;.\;.\;.\; a_{n\cdot n}) \qquad \text{(VII. 5).}$$

In this definition the rule of signs and the rule of term-formation are inseparable—a peculiarity already observed in the case of Bézout's rule of 1764.

After the definitions various technical terms are introduced. The n^2 different quantities involved in

$$S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} \;.\;.\;.\;.\; a_{n\cdot n})$$

are arranged thus

$$\left\{\begin{array}{lllll} a_{1\cdot1}, & a_{1\cdot2}, & a_{1\cdot3}, & .\;.\;.\;. & a_{1\cdot n} \\ a_{2\cdot1}, & a_{2\cdot2}, & a_{2\cdot3}, & .\;.\;.\;. & a_{2\cdot n} \\ a_{3\cdot1}, & a_{3\cdot2}, & a_{3\cdot3}, & .\;.\;.\;. & a_{3\cdot n} \\ \&c. \;.\;.\;.\;.\;. & & & & \\ a_{n\cdot1}, & a_{n\cdot2}, & a_{n\cdot3}, & .\;.\;.\;. & a_{n\cdot n} \end{array}\right.$$

"sur un nombre égal à n de lignes horizontales et sur autant de colonnes verticales," and as thus arranged are said to form a *symmetric system* of order n. The individual quantities $a_{1\cdot1}$, &c. are called the *terms* of the system, and the letter a when free of suffixes the *characteristic*. The "terms" in a horizontal line are said to form a *suite horizontale*, in a vertical column a *suite verticale*. *Conjugate* terms are defined as those whose suffixes ("indices") differ in order, *e.g.*, $a_{2\cdot3}$ and $a_{3\cdot2}$; and terms which are self-conjugate, *e.g.*, $a_{1\cdot1}$, $a_{2\cdot2}$, . . . are called *principal* terms. The determinant is said to *belong* to the system, or to be the determinant *of* the system. The parts of the expanded determinant which are connected by the signs + and − are called *symmetric products*, and the product

$$a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} \;.\;.\;.\;.\; a_{n\cdot n}$$

of the principal "terms" is called the *principal* product. The "principal product," however, is also called the *terme indicatif* of the determinant, and thus an awkward double use of the word "terme" is brought into prominence. The system

$$\begin{cases} a_{1\cdot1} & a_{2\cdot1} & a_{3\cdot1} & \ldots\ldots & a_{n\cdot1} \\ a_{1\cdot2} & a_{2\cdot2} & a_{3\cdot2} & \ldots\ldots & a_{n\cdot2} \\ a_{1\cdot3} & a_{2\cdot3} & a_{3\cdot3} & \ldots\ldots & a_{n\cdot3} \\ \&c. \ldots\ldots \\ a_{1\cdot n} & a_{2\cdot n} & a_{3\cdot n} & \ldots\ldots & a_{n\cdot n} \end{cases}$$

derived from the previous system by interchanging the suffixes of each "terme" is said to be *conjugate* to the previous system. A symbol for each of these systems is got by taking the last "terme" of its first "suite horizontale," and enclosing the "terme" in brackets: in this way we are enabled to say that *$(a_{1\cdot n})$ and $(a_{n\cdot 1})$ are conjugate systems.*

In the course of these explanations a modification of the rule of term-formation is incidentally noted, the form taken being specially applicable when the quantities of the system have been disposed in a square. Cauchy's wording of this now familiar rule is (p. 55)—

> . . . . . "pour former chacun des termes dont il s'agit, il suffira de multiplier entre elles n quantités différentes prises respectivement dans les différentes colonnes verticales du système, et situées en même temps dans les diverses lignes horizontales de ce système." (II. 6).

Here we may note in passing that the disposal of the "termes" in a square might have enabled Cauchy to point out (which he did not do) the difference between Gauss' use of the word "determinant" and his own, by saying that the "determinant of a form" had its conjugate "termes" equal.

The rule of signs applicable to alternating functions in general is modified for the special case of determinants, and takes the following form (p. 56):—

> "Étant donné un produit symétrique quelconque, pour obtenir le signe dont il est affecté dans le déterminant
>
> $$S(\pm a_{1\cdot1}a_{2\cdot2}a_{3\cdot3} \ldots\ldots a_{n\cdot n})$$
>
> il suffira d'appliquer la règle qui sert à déterminer le signe d'un terme pris à volonté dans une fonction symétrique alternée. Soit
>
> $$a_{\alpha\cdot1}\, a_{\beta\cdot2} \ldots\ldots a_{\zeta\cdot n}$$
>
> le produit symétrique dont il s'agit, et désignons par g le

nombre des substitutions circulaires équivalentes à la substitution

$$\begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ \alpha & \beta & \gamma & \ldots\ldots & \zeta \end{pmatrix}.$$

Ce produit devra être affecté du signe $+$, si $n-g$ est un nombre pair, et du signe $-$ dans le cas contraire." (III. 18).

Thus if the sign of the term

$$a_{6\cdot1}\,a_{8\cdot2}\,a_{3\cdot3}\,a_{1\cdot4}\,a_{9\cdot5}\,a_{2\cdot6}\,a_{5\cdot7}\,a_{4\cdot8}\,a_{7\cdot9}$$

in the determinant

$$\mathrm{S}(\pm a_{1\cdot1}\,a_{2\cdot2}\,a_{3\cdot3}\,\ldots\ldots\,a_{9\cdot9}),$$

be wanted, we write the series of first suffixes 6, 8, . . . under the corresponding suffixes of the "principal product," that is to say, under the series 1, 2, 3 . . . 9, obtaining the interchange

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 8 & 3 & 1 & 9 & 2 & 5 & 4 & 7 \end{pmatrix};$$

this we separate into circular interchanges, finding them three in number, viz.,

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix},\ \begin{pmatrix} 5 & 7 & 9 \\ 9 & 5 & 7 \end{pmatrix},\ \begin{pmatrix} 1 & 2 & 4 & 6 & 8 \\ 6 & 8 & 1 & 2 & 4 \end{pmatrix};$$

and the determinant being of the 9th order, we thence conclude that the desired sign is $(-)^{9-3}$, *i.e.*, $+$. In connection with this subject a modification of Cramer's rule is given, no reference being made to "dérangements" at all. Put into the fewest possible words it is—
The sign of the term $a_{\alpha\cdot1}\,a_{\beta\cdot2}\,\ldots\ldots\,a_{\zeta\cdot n}$ *is the same as the sign of the difference-product of the first suffixes, that is, the sign of*

$$(\beta-\alpha)(\gamma-\alpha)\ldots\ldots(\zeta-\alpha)(\gamma-\beta)\ldots\ldots \qquad \text{(III. 19)}.$$

For example, the sign of

$$a_{6\cdot1}\,a_{8\cdot2}\,a_{3\cdot3}\,a_{1\cdot4}\,a_{9\cdot5}\,a_{2\cdot6}\,a_{5\cdot7}\,a_{4\cdot8}\,a_{7\cdot9},$$

above sought, is the sign of the difference-product of

6, 8, 3, 1, 9, 2, 5, 4, 7

i.e., the sign of

$$\begin{array}{r} (7-4)(7-5)(7-2)(7-9)(7-1)(7-3)(7-8)(7-6) \\ \times\ (4-5)(4-2)\ \ldots\ldots\ldots\ldots\ (4-6) \\ \times\ (5-2)\ \ldots\ldots\ldots\ldots\ (5-6) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \times\ (8-6) \end{array}$$

The object which Cauchy had in view in stating the rule in this unnecessarily complex form was doubtless to show its essential identity with the rule implied in his new definition. He says (p. 58)—

"On démontre facilement cette règle par ce qui précède, attendu qu'une transposition opérée entre deux indices change toujours, comme on l'a fait voir, le signe du produit

$$(a_\beta - a_\alpha)\,(a_\gamma - a_\alpha)\;.\;.\;.\;.\;(a_\zeta - a_\alpha)\,(a_\gamma - a_\beta)\;.\;.\;.\;.\;,$$

et par conséquent celui du produit

$$(\beta - \alpha)\,(\gamma - \alpha)\;.\;.\;.\;(\zeta - \alpha)\,(\gamma - \beta)\;.\;.\;.\;.\text{"}$$

The way having thus been prepared, the propositions of determinants are entered on. Those known to his predecessors we may dispose of rapidly, giving little, if anything, more than the enunciation of them, in order that the new garb in which they appear may be seen.

. . . . "le déterminant du système $(a_{n\cdot 1})$ est égal à celui du système $(a_{1\cdot n})$ . . . . . En conséquence, dans l'expression

$$S(\pm a_{1\cdot 1}a_{2\cdot 2}\;.\;.\;.\;.\;a_{n\cdot n})$$

on peut supposer indifféremment, ou que le signe S se rapporte aux premiers indices, ou qu'il se rapporte aux seconds: (IX. 2).

Si l'on échange entre elles deux suites horizontales ou deux suites verticales du système $(a_{1\cdot n})$ de manière à faire passer dans une des suites tous les termes de l'autre et réciproquement on obtiendra un nouveau système symétrique, dont le déterminant sera évidemment égal mais de signe contraire à celui du système $(a_{1\cdot n})$. Si l'on répète la même opération plusieurs fois de suite, on obtiendra divers systèmes symétriques dont les déterminans seront égaux entre eux, mais alternativement positifs et négatifs. On peut faire la même remarque à l'égard du systéme $(a_{n\cdot 1})$ (XI. 3).

. . . . . si l'on développe la fonction symétrique alternée

$$S[\pm a_{n\cdot n}S(\pm a_{1\cdot 1}\,a_{2\cdot 2}\;.\;.\;.\;.\;a_{n-1\cdot n-1})]$$

tous les termes du développement seront des produits symétriques de l'ordre n, qui auront l'unité pour coefficient. Ces

termes seront donc respectivement égaux à ceux qu'on obtient en développant le déterminant

$$D_n = S(\pm a_{1\cdot 1}\, a_{2\cdot 2} \;.\;.\;.\;.\; a_{n\cdot n})\,;$$

et comme le produit principal $a_{1\cdot 1}\, a_{2\cdot 2} \;.\;.\;.\; a_{n\cdot n}$ est positif de part et d'autre, on aura nécessairement

$$\begin{aligned} D_n &= S[\pm a_{n\cdot n} S(\pm a_{1\cdot 1}\, a_{2\cdot 2} \;.\;.\;.\;.\; a_{n-1\cdot n-1})] \qquad (\text{VI. } 3)\\ &= a_{n\cdot n} b_{n\cdot n} + a_{n-1\cdot n} b_{n-1\cdot n} + \;.\;.\;.\;.\; + a_{1\cdot n} b_{1\cdot n}\,. \end{aligned}$$

En général, si l'on désigne par μ l'un des indices $1, 2, 3, \ldots, n$ on trouvera de la même manière

$$D_n = S[\pm a_{\mu\cdot\mu} S(\pm a_{1\cdot 1}\, a_{2\cdot 2} \;.\;.\;.\;.\; a_{\mu-1\cdot\mu-1} a_{\mu+1\cdot\mu+1} \;.\;.\;.\; a_{n\cdot n})] \qquad (\text{VI. } 4).$$

. . . . . Cette dernière équation

$$0 = a_{1\cdot\nu} b_{1\cdot\mu} + a_{2\cdot\nu} b_{2\cdot\mu} + \;.\;.\;.\; + a_{n\cdot\nu} b_{n\cdot\mu} \qquad (\text{XII. } 6)$$

sera satisfaite toutes les fois que ν et μ seront deux nombres différens l'un de l'autre.

. . . . on aura donc aussi

$$\begin{aligned} D_n &= a_{\mu\cdot 1} b_{\mu\cdot 1} + a_{\mu\cdot 2} b_{\mu\cdot 2} + \;.\;.\;.\;.\; + a_{\mu\cdot n} b_{\mu\cdot n} \qquad (\text{VI. } 4)\\ 0 &= a_{\nu\cdot 1} b_{\mu\cdot 1} + a_{\nu\cdot 2} b_{\mu\cdot 2} + \;.\;.\;.\;.\; + a_{\nu\cdot n} b_{\mu\cdot n} \qquad (\text{XII. } 6) \end{aligned}$$

les indices μ et ν étant censés inégaux."

The expressions here denoted by $b_{1\cdot 1}$, $b_{1\cdot 2}$, ... are spoken of as *adjugate* ("adjointes") to $a_{1\cdot 1}$, $a_{1\cdot 2}$, . . . ; and the system

$$\begin{cases} b_{1\cdot 1} & b_{1\cdot 2} \;.\;.\;.\;.\;.\;. \; b_{1\cdot n} \\ b_{2\cdot 1} & b_{2\cdot 2} \;.\;.\;.\;.\;.\;. \; b_{2\cdot n} \\ \&c. & .\;.\;.\;.\;.\;.\;.\;.\;. \\ b_{n\cdot 1} & b_{n\cdot 2} \;.\;.\;.\;.\;.\;. \; b_{n\cdot n} \end{cases}$$

as adjugate to the system $(a_{1\cdot n})$. Similarly the system $(b_{n\cdot 1})$ is said to be adjugate to the system $(a_{n\cdot 1})$; and, on the other hand, it is said to be *adjugate and conjugate* to the system $(a_{1\cdot n})$.

Up to this point no new property has been brought forward. The following paragraph (p. 68), however, opens new ground, the formula given in it being of some considerable importance in the after development of the theory.

"Si dans le système de quantités $(a_{1\cdot n})$ on supprime la dernière

suite horizontale et la dernière suite verticale, on aura le système suivant,

$$\begin{cases} a_{1\cdot1}, & a_{2\cdot1} \ldots\ldots & a_{1\cdot n-1}, \\ a_{2\cdot1}, & a_{2\cdot2} & a_{2\cdot n-1}, \\ \&c. \ldots\ldots & & \\ a_{n-1\cdot1}, & a_{n-1\cdot2} & a_{n-1\cdot n-1}, \end{cases}$$

que je désignerai à l'ordinaire par $(a_{1\cdot n-1})$.

"Soit maintenant $(e_{1\cdot n-1})$ le système adjoint au précédent. Si dans l'équation (13) on change b en e et n en $n-1$, on aura en général

$$D_{n-1} = b_{n\cdot n} = a_{\mu\cdot1}e_{\mu\cdot1} + a_{\mu\cdot2}e_{\mu\cdot2} + \ldots + a_{\mu\cdot n-1}e_{\mu\cdot n-1}\,.$$

Pour déduire de cette dernière équation la valeur de $b_{\mu\cdot n}$, il suffira en vertu des règles établies, de changer $a_{\mu\cdot\nu}$ en $a_{n\cdot\nu}$ dans l'expression précédente de $b_{n\cdot n}$, et de changer en outre le signe du second membre : ou aura donc généralement

$$b_{\mu\cdot n} = -\left(a_{n\cdot1}e_{\mu\cdot1} + a_{n\cdot2}e_{\mu\cdot2} + \ldots\ldots + a_{n\cdot n-1}e_{\mu\cdot n-1}\right).$$

Si dans cette équation on donne successivement à μ toutes les valeurs entières depuis 1 jusqu'à $n-1$, et que l'on substitue les valeurs qui en résulteront pour $b_{1\cdot n}$, $b_{2\cdot n}, \ldots b_{n-1\cdot n}$ dans l'équation

$$D_n = a_{1\cdot n}b_{1\cdot n} + a_{2\cdot n}b_{2\cdot n} + \ldots\ldots + a_{n\cdot n}b_{n\cdot n},$$

on obtiendra la formule suivante,

$$D_n = a_{n\cdot n}b_{n\cdot n} - \begin{cases} a_{1\cdot n}a_{n\cdot1}e_{1\cdot1} & + a_{2\cdot n}a_{n\cdot2}e_{2\cdot2} + \ldots + a_{n-1\cdot n}a_{n\cdot n-1}e_{n-1\cdot n-1} \\ + a_{1\cdot n}(a_{n\cdot2}e_{1\cdot2} & + a_{n\cdot3}e_{1\cdot3} \quad + \ldots + a_{n\cdot n-1}e_{1\cdot n-1}) \\ + a_{2\cdot n}(a_{n\cdot1}e_{2\cdot1} & + a_{n\cdot3}e_{2\cdot3} \quad + \ldots + a_{n\cdot n-1}e_{2\cdot n-1}) \\ + \&c. \ldots\ldots & \\ + a_{n-1\cdot n}(a_{n\cdot1}e_{n-1\cdot1} + a_{n\cdot2}e_{n-1\cdot2} + \ldots + a_{n\cdot n-2}e_{n-1\cdot n-2}). & \end{cases}$$

Cette équation peut être mise sous la forme

$$D_n = a_{n\cdot n}D^*_{n\cdot1} - S^{n-1}S^{n-1}(a_{\nu\cdot n}a_{n\cdot\mu}e_{\nu\cdot\mu}), \qquad \text{(XXXVII.)}$$

les deux signes S étant relatifs le premier à l'indice μ et le second à l'indice ν."

This is the well-known formula nowadays described as giving

* Misprint in original, for D_{n-1}.

the development of a determinant according to binary products of a row and column. The special row here used is the n^{th} and the special column the n^{th} likewise.

The four pages regarding the application of determinants to the solution of a set of simultaneous equations may be passed over with the remark that they give evidence of the importance attached by Cauchy to his new definition of determinants, the solution in the case of the example

$$\left.\begin{array}{l} a_1x_1 + b_1x_2 = m_1 \\ a_2x_1 + b_2x_2 = m_2 \end{array}\right\}$$

being first put in the form

$$x = \frac{mb(b-m)}{ab(b-a)}, \quad y = \frac{am(m-a)}{ab(b-a)};$$

and similarly in the case of the example

$$a_rx_1 + b_rx_2 + c_rx_3 = m_r \qquad (r = 1, 2, 3).$$

The determinant solution of a set of simultaneous equations is put to good use by Cauchy to obtain new properties of the functions. Taking the set of equations

$$(20)\left\{\begin{array}{l} a_{1\cdot1}x_1 + a_{1\cdot2}x_2 + \ldots\ldots + a_{1\cdot n}x_n = m_1 \\ a_{2\cdot1}x_1 + a_{2\cdot2}x_2 + \ldots\ldots + a_{2\cdot n}x_n = m_2 \\ \&c. \ldots\ldots \\ a_{n\cdot1}x_1 + a_{n\cdot2}x_2 + \ldots\ldots + a_{n\cdot n}x_n = m_n \end{array}\right.$$

and solving for $x_1, x_2, \ldots$ he obtains of course the set

$$\left.\begin{array}{l} m_1b_{1\cdot1} + m_2b_{2\cdot1} + \ldots\ldots + m_nb_{n\cdot1} = D_nx_1, \\ m_1b_{1\cdot2} + m_2b_{2\cdot2} + \ldots\ldots + m_nb_{n\cdot2} = D_nx_2, \\ \&c. \ldots\ldots\ldots \\ m_1b_{1\cdot n} + m_2b_{2\cdot n} + \ldots\ldots + m_nb_{n\cdot n} = D_nx_n, \end{array}\right\}$$

where $b_{1\cdot1}, b_{2\cdot1}, \ldots\ldots$ have the signification above indicated, and D_n stands for $S(\pm a_{1\cdot1}a_{2\cdot2} \ldots a_{n\cdot n})$. This second set may be treated in the same way as the first set, the quantities $m_1, m_2, \ldots, m_n$ being viewed as the unknowns. To express the result the system of quantities adjugate to $(b_{1\cdot n})$ is denoted by $(c_{1\cdot n})$, and the determinant of the system $(b_{1\cdot n})$ is denoted by B_n, the new set thus being

$$(27)\quad\begin{cases} c_{1\cdot1}D_n x_1 + c_{1\cdot2}D_n x_2 + \ldots\ldots + c_{1\cdot n}D_n x_n = B_n m_1, \\ c_{2\cdot1}D_n x_1 + c_{2\cdot2}D_n x_2 + \ldots\ldots + c_{2\cdot n}D_n x_n = B_n m_2, \\ \&c. \ldots\ldots\ldots \\ c_{n\cdot1}D_n x_1 + c_{n\cdot2}D_n x_2 + \ldots\ldots + c_{n\cdot n}D_n x_n = B_n m_n, \end{cases}$$

Cauchy then proceeds (p. 77)—

"Les équations (27) peuvent encore être mises sous la forme suivante,

$$\begin{cases} c_{1\cdot1}\frac{D_n}{B_n}x_1 + c_{1\cdot2}\frac{D_n}{B_n}x_2 + \ldots\ldots + c_{1\cdot n}\frac{D_n}{B_n}x_n = m_1, \\ c_{2\cdot1}\frac{D_n}{B_n}x_1 + c_{2\cdot2}\frac{D_n}{B_n}x_2 + \ldots\ldots + c_{2\cdot n}\frac{D_n}{B_n}x_n = m_2, \\ \&c. \ldots\ldots\ldots \\ c_{n\cdot1}\frac{D_n}{B_n}x_1 + c_{n\cdot2}\frac{D_n}{B_n}x_2 + \ldots\ldots + c_{n\cdot n}\frac{D_n}{B_n}x_n = m_n; \end{cases}$$

et comme celles-ci doivent avoir lieu en même temps que les équations (20), sans que l'on suppose d'ailleurs entre les termes de la suite $x_1, x_2, \ldots\ldots, x_n$ et ceux du système $(a_{1\cdot n})$ aucune relation particulière, il faudra nécessairement que l'on ait, quels que soient μ et ν,

$$c_{\mu\cdot\nu}\frac{D_n}{B_n} = a_{\mu\cdot\nu},$$

ou

$$c_{\mu\cdot\nu} = \frac{B_n}{D_n}a_{\mu\cdot\nu}. \qquad \text{(XXXVIII.)}$$

Cette équation établit un rapport constant entre les termes du système $(a_{1\cdot n})$ et les termes du système adjoint du second ordre $(c_{1\cdot n})$."

More definitely, and in more modern nomenclature, the theorem is

The ratio of any element of a determinant to the corresponding element of the second adjugate determinant is equal to the ratio of the determinant itself to its first adjugate. (XXXVIII.)

Attention is next directed to the group of equations—

$$\left.\begin{array}{lll}
a_{1\cdot1}a_{1\cdot1}+a_{1\cdot2}a_{1\cdot2}+\ldots+a_{1\cdot n}a_{1\cdot n}=m_{1\cdot1} & a_{2\cdot1}a_{1\cdot1}+a_{2\cdot2}a_{1\cdot2}+\ldots+a_{2\cdot n}a_{1\cdot n}=m_{1\cdot2}, \;\ldots & a_{n\cdot1}a_{1\cdot1}+a_{n\cdot2}a_{1\cdot2}+\ldots+a_{n\cdot n}a_{1\cdot n}=m_{1\cdot n} \\
a_{1\cdot1}a_{2\cdot1}+a_{1\cdot2}a_{2\cdot2}+\ldots+a_{1\cdot n}a_{2\cdot n}=m_{2\cdot1} & a_{2\cdot1}a_{2\cdot1}+a_{2\cdot2}a_{2\cdot2}+\ldots+a_{2\cdot n}a_{2\cdot n}=m_{2\cdot2}, \;\ldots & a_{n\cdot1}a_{2\cdot1}+a_{n\cdot2}a_{2\cdot2}+\ldots+a_{n\cdot n}a_{2\cdot n}=m_{2\cdot n} \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
a_{1\cdot1}a_{n\cdot1}+a_{1\cdot2}a_{n\cdot2}+\ldots+a_{1\cdot n}a_{n\cdot n}=m_{n\cdot1} & a_{2\cdot1}a_{n\cdot1}+a_{2\cdot2}a_{n\cdot2}+\ldots+a_{2\cdot n}a_{n\cdot n}=m_{n\cdot2}, \;\ldots & a_{n\cdot1}a_{n\cdot1}+a_{n\cdot2}a_{n\cdot2}+\ldots+a_{n\cdot n}a_{n\cdot n}=m_{n\cdot n}
\end{array}\right\}$$

Here there are three symmetric systems of quantities

$$(a_{1\cdot n}),\ (\alpha_{1\cdot n}),\ (m_{1\cdot n}),$$

the first appearing in every column of equations, the second in every row, and the third only once. The determinants of these systems are denoted by

$$D_n,\ \delta_n,\ M_n,$$

respectively: that is to say

$$D_n = S(\pm\, a_{1\cdot 1}\, a_{2\cdot 2}\ .\ .\ .\ a_{n\cdot n})$$
$$\delta_n = S(\pm\, \alpha_{1\cdot 1}\, \alpha_{2\cdot 2}\ .\ .\ .\ \alpha_{n\cdot n})$$
$$M_n = S(\pm\, m_{1\cdot 1} m_{2\cdot 2}\ .\ .\ .\ m_{n\cdot n})\,.$$

If now in

$$S(\pm\, a_{1\cdot 1} a_{2\cdot 2}\ .\ .\ .\ a_{n\cdot n})$$

there be substituted for $m_{1\cdot 1}$, $m_{1\cdot 2}$, ... their values as given by the group of equations, there will be obtained a function of all the a's and α's, which must be an alternating function with respect to the first indices of the a's and also with respect to the first indices of the α's. Further, since each of the m's is of the first degree in the a's and of the first degree also in the α's, each term of the development of $S(\pm\, m_{1\cdot 1} m_{2\cdot 2}\ .\ .\ .\ .\ m_{n\cdot n})$ must evidently be of the form

$$\pm\, a_{1\cdot \mu} a_{2\cdot \nu}\ .\ .\ .\ .\ a_{n\cdot \pi} \alpha_{1\cdot \mu_1} \alpha_{2\cdot \nu_1}\ .\ .\ .\ .\ \alpha_{n\cdot \pi}\,.$$

But the development by reason of its double alternating character cannot contain such a term without containing all the terms of the product

$$\pm\, S(\pm\, a_{1\cdot \mu} a_{2\cdot \nu}\ .\ .\ .\ a_{n\cdot \pi}) S(\pm\, \alpha_{1\cdot \mu_1} \alpha_{2\cdot \nu_1}\ .\ .\ .\ \alpha_{n\cdot \pi_1})\,.$$

Consequently it must equal one or more products of this kind. But again the indices μ, ν, . . ., π are either all different or not. If they be different, we have

$$S(\pm\, a_{1\cdot \mu} a_{2\cdot \nu}\ .\ .\ .\ a_{n\cdot \pi}) = \pm\, S(\pm\, a_{1\cdot 1} a_{2\cdot 2}\ .\ .\ .\ a_{n\cdot n}) = \pm\,\delta\ ;$$

and if any two of them be equal

$$S(\pm\, a_{1\cdot \mu} a_{2\cdot \nu}\ .\ .\ .\ a_{n\cdot \pi}) = 0\,.$$

The like is true in regard to $S\pm(\alpha_{1\cdot \mu_1}, \alpha_{2\cdot \nu_1}\ .\ .\ .\ \alpha_{n\cdot \pi_1})$. This enables us to conclude that the development of M_n is equal to one or more products of the form

$$\pm\, D_n \delta_n :$$

in other words, that

$$M_n = c D_n \delta_n\,,$$

where c is a constant. But if we take the very special case where

$$a_{\mu\cdot\mu}=1,\quad a_{\mu\cdot\mu}=1,\quad a_{\mu\cdot\nu}=0,\quad a_{\mu\cdot\nu}=0,$$

and where consequently

$$m_{\mu\cdot\mu}=1,\quad m_{\mu\cdot\nu}=0\,,$$

we see that

$$\mathrm{M}_n=1,\quad \mathrm{D}_n=1,\quad \delta_n=1\,,$$

and that therefore

$$c=1\,.$$

Hence the final result is

$$\mathrm{M}_n=\mathrm{D}_n\delta_n\,. \qquad\qquad (\text{xvii. } 5).$$

This, the now well-known multiplication-theorem of determinants, Cauchy puts in words as follows (p. 82) :—

> *Lorsqu'un système de quantités est déterminé symétriquement au moyen de deux autres systèmes, le déterminant du système résultant est toujours égal au produit des déterminans des deux systèmes composans.* (XVII. 5).

It is quite clear, from what has been said above, that it was discovered independently, and about the same time, by Binet and Cauchy, and ought to bear the names of both. Binet has the further merit of having reached a theorem of which Cauchy's is a special case, and then made an additional generalisation in a different direction; and Cauchy has the advantage over Binet of having produced, along with his special case, a satisfactory proof of it.

From the theorem Cauchy goes on to deduce several results equally important. Substituting for the system $(a_{1\cdot n})$ the system $(b_{1\cdot n})$ adjugate to $(a_{1\cdot n})$ so that

$$\delta_n=\mathrm{S}(\pm b_{1\cdot 1}b_{2\cdot 2}\ .\ .\ .\ b_{n\cdot n})=\mathrm{B}_n\,,$$

we know that then

$$m_{\mu\cdot\mu}=\mathrm{D}_n \text{ and } m_{\mu\cdot\nu}=0\,;$$

that consequently M_n consists of but a single term, viz.

$$m_{1\cdot 1}m_{2\cdot 2}\ .\ .\ .\ m_{n\cdot n},\ \ i.e.\ \mathrm{D}_n^n\,:$$

and that therefore by the theorem

$$\mathrm{D}_n^n=\mathrm{B}_n\mathrm{D}_n\,,$$

whence

$$\mathrm{B}_n=\mathrm{D}_n^{n-1}\,. \qquad\qquad (\text{xxi. } 2).$$

This result, afterwards so well known, Cauchy translates into words as follows (p. 82):—

> . . . . *le déterminant du système* $(b_{1\cdot n})$ *adjoint au système* $(a_{1\cdot n})$ *est égal à la* $(n-1)^{me}$ *puissance du déterminant de ce dernier système.* (XXI. 2).

Again, by returning to the identity,

$$c_{\mu\cdot\nu} = \frac{B_n}{D_n} a_{\mu\cdot\nu}$$

and substituting the value of B_n just obtained, there is deduced the result

$$c_{\mu\cdot\nu} = D_n^{n-2} a_{\mu\cdot\nu}; \qquad \text{(XXXIX.)}$$

or, in words,

> . . . *étant donné un terme quelconque* $a_{\mu\cdot\nu}$ *du système* $(a_{1\cdot n})$, *pour obtenir le terme correspondant du système adjoint du second ordre* $(c_{1\cdot n})$ *il suffira de multiplier le terme donné par la* $(n-2)^{me}$ *puissance du déterminant du premier système.* (XXXIX.)

A considerable amount of space (pp. 82–92) is devoted to the consideration of the adjugate systems of

$$(a_{1\cdot n}), \quad (a_{1\cdot n}), \quad (m_{1\cdot n}),$$

and the adjugates of these adjugates; but nothing new is elicited. The section closes with the manifest identity

$$\begin{aligned} &(a_{1\cdot 1} + a_{2\cdot 1} + \ldots + a_{n\cdot 1})\,(a_{1\cdot 1} + a_{2\cdot 1} + \ldots + a_{n\cdot 1}) \\ &+ (a_{1\cdot 2} + a_{2\cdot 2} + \ldots + a_{n\cdot 2})\,(a_{1\cdot 2} + a_{2\cdot 2} + \ldots + a_{n\cdot 2}) \\ &+ \&c. \ldots\ldots \\ &+ (a_{1\cdot n} + a_{2\cdot n} + \ldots + a_{n\cdot n})\,(a_{1\cdot n} + a_{2\cdot n} + \ldots + a_{n\cdot n}) \\ = {} & m_{1\cdot 1} + m_{2\cdot 1} + \ldots + m_{n\cdot 1} \\ &+ m_{1\cdot 2} + m_{2\cdot 2} + \ldots + m_{n\cdot 2} \\ &+ \ldots\ldots \\ &+ m_{1\cdot n} + m_{2\cdot n} + \ldots + m_{n\cdot n}, \end{aligned}$$

which, using later technical terms, we may express as follows:—

If there be two determinants, and the sum of the elements of one first column be multiplied by the sum of the elements of the other first column, the sum of the elements of one second column by the sum of the elements of the other second column, and so on, then the sum of these products is equal to the sum of the elements of the product of the two determinants. (XL.)

The third section breaks entirely fresh ground, its heading being

Des Systèmes de Quantités dérivées et de leurs Déterminans.

Of the integers 1, 2, 3, . . ., n all the possible sets of p integers are supposed to be taken, and arranged in order on the principle that any one has precedence of any other if the product of the members of the former be less than the product of the members of the latter. The number $n(n-1)\ldots\ldots(n-p+1)\,/\,1.\,2.\,3\,\ldots\ldots p$ of the said sets being denoted by P, the P[th] and last set would thus be

$$n-p+1,\ n-p+2,\ \ldots\ldots,\ n-1,\ n.$$

Now, any two of the sets being fixed upon, say the μ^{th} and ν^{th}, the system of quantities $(a_{1\cdot n})$ is returned to, and from it are deleted (1) all the "termes" whose first index is not found in the μ^{th} set, and (2) all the "termes" whose second index is not found in the ν^{th} set. What is left after this action is clearly "un système de quantités symétriques de l'ordre p," the determinant of which may be denoted by $a^{(p)}_{\mu,\nu}$. For example, if $\mu=\nu=1$, all the a's would be deleted whose first or second index was not included in the set 1, 2, 3, . . ., p, and there would be left the system

$$\left\{\begin{matrix} a_{1\cdot1} & a_{1\cdot2} & \ldots\ldots & a_{1\cdot p} \\ a_{2\cdot1} & a_{2\cdot2} & \ldots\ldots & a_{2\cdot p} \\ \&c. \ldots\ldots & & & \\ a_{p\cdot1} & a_{p\cdot2} & \ldots\ldots & a_{p\cdot p} \end{matrix}\right.$$

of which the determinant would be denoted by

$$a^{(p)}_{1\cdot1}.$$

As any one of the P sets could be taken along with any other, preparatory to forming such a determinant, there would necessarily be in all P × P possible determinants. Arranged in a square as follows:—

$$\left\{\begin{matrix} a^{(p)}_{1\cdot1} & a^{(p)}_{1\cdot2} & \ldots\ldots & a^{(p)}_{1\cdot \mathrm{P}} \\ a^{(p)}_{2\cdot1} & a^{(p)}_{2\cdot2} & \ldots\ldots & a^{(p)}_{2\cdot \mathrm{P}} \\ \&c. \ldots\ldots & & & \\ a^{(p)}_{\mathrm{P}\cdot1} & a^{(p)}_{\mathrm{P}\cdot2} & \ldots\ldots & a^{(p)}_{\mathrm{P}\cdot \mathrm{P}} \end{matrix}\right.$$

they manifestly form "un système symétrique de l'ordre P," which, in strict accordance with previous convention, is denoted by

$$\left(a_{1\cdot P}^{(p)}\right).$$

Cauchy then proceeds (p. 96)—

Si l'on donne successivement à p toutes les valeurs

$$1, 2, 3, \ldots, n-3, n-2, n-1$$

P prendra les valeurs suivantes,

$$n, \frac{n(n-1)}{1.2}, \frac{n(n-1)(n-2)}{1.2.3}, \ldots, \frac{n(n-1)}{1.2}, n,$$

et l'on obtiendra par suite un nombre égal à n − 1 de systèmes symétriques différens les uns des autres, dont le premier sera le système donné $(a_{1\cdot n})$. Ces différens systèmes seront désignés respectivement par

$$(a_{1\cdot n}), \left[a_{1\cdot\frac{n(n-1)}{1.2}}^{(2)}\right], \left[a_{1\cdot\frac{n(n-1)(n-2)}{1.2.3}}^{(3)}\right], \ldots\ldots$$
$$\left[a_{1\cdot\frac{n(n-1)(n-2)}{1.2.3}}^{(n-3)}\right], \left[a_{1\cdot\frac{n(n-1)}{1.2}}^{(n-2)}\right], \left(a_{1\cdot n}^{(n-1)}\right);$$

je les appellerai *systèmes dérivés* de $(a_{1\cdot n})$. Parmi ces systèmes, ceux qui correspondent à des valeurs de p dont la somme est égale à n sont toujours de même ordre; je les appellerai *systèmes dérivés complémentaires.* Ainsi en général

$$\left(a_{1\cdot P}^{(p)}\right) \text{ et } \left(a_{1\cdot P}^{(n-p)}\right)$$

sont deux systèmes dérivés complémentaires l'un de l'autre, dont l'ordre est égal a

$$P = \frac{n(n-1)\ldots(n-p+1)}{1.2.3\ldots p}.$$

Up to this point a thorough understanding of the notation

$$\left(a_{1\cdot P}^{(p)}\right)$$

is the one essential. Taking the particular instance

$$\left(a_{1\cdot 10}^{(2)}\right)$$

we first call to mind that it is an abbreviation for the "système symétrique" whose first row has for its last "terme" the determinant

$$a_{1\cdot 10}^{(2)}$$

—that is to say, an abbreviation for the system whose determin-

ant we should nowadays write in the form

$$\begin{vmatrix} a^{(2)}_{1.1} & a^{(2)}_{1.2} & \cdots & a^{(2)}_{1.10} \\ a^{(2)}_{2.1} & a^{(2)}_{2.2} & \cdots & a^{(2)}_{2.10} \\ \cdot & \cdot & \cdots & \cdot \\ a^{(2)}_{10.1} & a^{(2)}_{10.2} & \cdots & a^{(2)}_{10.10} \end{vmatrix}.$$

The next point is to realise what determinants are denoted by

$$a^{(2)}_{1.1},\quad a^{(2)}_{1.2},\ \ldots\ldots\ldots$$

Now the number 10 being of necessity a combinatorial, and, as the figure in brackets above it indicates, of the form

$$\frac{n(n-1)}{1\,.\,2},$$

we see that n must be 5, and that the said determinants are all derived from

$$\begin{vmatrix} a_{1\cdot1} & a_{1\cdot2} & a_{1\cdot3} & a_{1\cdot4} & a_{1\cdot5} \\ a_{2\cdot1} & a_{2\cdot2} & a_{2\cdot3} & a_{2\cdot4} & a_{2\cdot5} \\ a_{3\cdot1} & a_{3\cdot2} & a_{3\cdot3} & a_{3\cdot4} & a_{3\cdot5} \\ a_{4\cdot1} & a_{4\cdot2} & a_{4\cdot3} & a_{4\cdot4} & a_{4\cdot5} \\ a_{5\cdot1} & a_{5\cdot2} & a_{5\cdot3} & a_{5\cdot4} & a_{5\cdot5} \end{vmatrix}.$$

The details of the process of derivation are recalled in connection with the interpretation of the pairs of suffixes. A requisite preliminary is to form all the different pairs of the numbers 1, 2, 3, 4, 5; arrange them in the order

12, 13, 14, 15, 23, 24, 25, 34, 35, 45;

and then number them

1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

These last are the numbers from which the suffixes are taken, and what each one as a suffix refers to, is the combination under which it is here placed. For example, the first suffix in $a^{(2)}_{1.1}$ refers to the combination 1 2, and implies the deletion of all the rows of the above determinant of the fifth order except the 1st and 2nd; the second suffix refers to the same combination, and implies the deletion of all the columns except the 1st and 2nd; and the symbol as a whole thus comes to stand for

$$\begin{vmatrix} a_{1\cdot1} & a_{1\cdot2} \\ a_{2\cdot1} & a_{2\cdot2} \end{vmatrix}. \qquad \text{(XLI. 2.)}$$

Interpreting $a^{(2)}_{1.2}$, $a^{(2)}_{1.3}$, ... in the same way, we see that

$$\left(a^{(2)}_{1.10}\right)$$

is a compact notation for the system of which the determinant is

$$\left|\begin{array}{cccc} |a_{1\cdot1}a_{2\cdot2}|, & \cdots\cdots & , & |a_{1\cdot4}a_{2\cdot5}| \\ \cdots & \cdots & \cdots & \cdots \\ \cdot\ \cdot & \cdots\cdots & , & |a_{3\cdot4}a_{4\cdot5}| \\ |a_{3\cdot1}a_{5\cdot2}| & ,\ |a_{3\cdot3}a_{5\cdot4}|, & |a_{3\cdot3}a_{5\cdot5}|, & |a_{3\cdot4}a_{5\cdot5}| \\ |a_{4\cdot1}a_{5\cdot2}|, & \cdot\ \cdot\ ,\ |a_{4\cdot3}a_{5\cdot4}|, & |a_{4\cdot3}a_{5\cdot5}|, & |a_{4\cdot4}a_{5\cdot5}| \end{array}\right|.$$

Similarly

$$\left(a^{(3)}_{1.10}\right)$$

stands for the system of which the determinant is

$$\left|\begin{array}{cccc} |a_{1\cdot1}a_{2\cdot2}a_{3\cdot3}|, & \cdots\cdots & , & |a_{1\cdot3}a_{2\cdot4}a_{3\cdot5}| \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots\cdots & , & |a_{2\cdot3}a_{3\cdot4}a_{5\cdot5}| \\ |a_{2\cdot1}a_{4\cdot2}a_{5\cdot3}|, & \cdot\ \cdot\ \cdot\ \cdot\ , & |a_{2\cdot2}a_{4\cdot4}a_{5\cdot5}|, & |a_{2\cdot3}a_{4\cdot4}a_{5\cdot5}| \\ |a_{3\cdot1}a_{4\cdot2}a_{5\cdot3}| & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , & |a_{3\cdot2}a_{4\cdot4}a_{5\cdot5}|, & |a_{3\cdot3}a_{4\cdot4}a_{5\cdot5}| \end{array}\right|.$$

and which is called the "complementary derived system." (XLI. 3.) To every "terme" of the latter there corresponds a "terme" of the former, the one "terme" consisting exactly of those a's of the original determinant which are awanting in the other. This relationship Cauchy goes on to mark by means of a name and a notation. He calls two such "termes," $|a_{3\cdot1}a_{4\cdot2}a_{5\cdot3}|$ and $|a_{1\cdot4}a_{2\cdot5}|$ for example, "termes complémentaires des deux systèmes;" (XLI. 4) and if the symbol for the one be by previous agreement

$$a^{(p)}_{\mu.\pi}$$

the symbol for the other is made *

$$a^{(n-p)}_{P-\mu+1\,.\,P-\pi+1} \qquad \text{(XLI. 5.)}$$

As for the signs of the "termes" in "derived systems," Cauchy's words are (p. 98)—

* If Cauchy had adopted a slightly different principle for determining the order of combinations, the μ^{th} combination of p things and the $(P-\mu+1)^{\text{th}}$ combination of $n-p$ things would have been mutually exclusive, and the convention here made in regard to notation would have been unnecessary.

'En général, il est facile de voir que le produit de deux termes complémentaires pris à volonté est toujours, au signe près, une portion de ce même déterminant (D_n). Cela posé, étant donné le signe de l'un de ces deux termes, on déterminera celui de l'autre par la condition que leur produit soit affecté du même signe que la portion correspondante du déterminant D_n."

All these preliminaries having been settled, the weighty matters of the section are entered on. The first of these is a complete and perfectly accurate statement of the expansion-theorem, known by the name of Laplace, but which, as we have seen, Laplace and even Bézout who followed him were very far from fully formulating. The passage is of the greatest interest. No better example could be chosen to illustrate the powerful grasp which Cauchy had of the subject. What Laplace and Bézout laboured at, lengthily expounding one special case after another, Cauchy sets forth with ease and in all its generality in the space of a page. His words are (p. 99)—

"On a fait voir dans le § 3ᵉ que la fonction symétrique alternée

$$S(\pm a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3}\ldots a_{n\cdot n}) = D_n$$

était équivalente à celle-ci

$$S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{n-1\cdot}\quad) \,.\, a_{n\cdot n}]\,.$$

On fera voir de même qu'elle est encore équivalente à

$$S[\pm S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}) \,.\, S(\pm a_{p+1\cdot p+1}\ldots a_{n-1\cdot n-1}a_{n\cdot n})]\,,$$

les opérations indiquées par le signe S pouvant être considérées comme relatives, soit aux premiers, soit aux seconds indices. On a d'ailleurs par ce qui précède

$$S(\pm a_{1\cdot 1}a_{2\cdot 2}\ldots a_{p\cdot p}) = \pm a^{(p)}_{1.1}\,,$$

$$S(\pm a_{p+1\cdot p+1}\ldots a_{n\cdot n}) = \pm a^{(n-p)}_{.\mathrm{P}}\,.$$

Enfin les signes des quantités de la forme

$$a^{(p)}_{1.1}\quad,\quad a^{(n-p)}_{\mathrm{P.P}}$$

doivent être tels que les produits semblables à

$$a^{(p)}_{1.1}a^{(n-p)}_{\mathrm{P.P}}$$

soient dans le déterminant D_n affectés du signe $+$. Cela posé, il résulte de l'équation

$$D_n = S[\pm S(\pm a_{1\cdot 1} a_{2\cdot 2} \ldots a_{p\cdot p}) \,.\, S(\pm a_{p+1\cdot p+1} \ldots a_{n\cdot n})],$$

que D_n est la somme de plusieurs produits de la forme

$$a_{1.1}^{(p)} \, a_{P.P}^{(n-p)}.$$

Selon que pour obtenir ces différens produits on échangera entre eux les premiers ou les seconds indices du système $(a_{1\cdot n})$, on trouvera ou l'équation

$$D_n = a_{1.1}^{p} \, a_{P.P}^{(n-p)} + a_{2.1}^{(p)} \, a_{P-1.P}^{(n-p)} + \qquad + a_{P.1}^{(p)} \, a_{1.P}^{(n-p)},$$

ou celle-ci

$$D_n = a_{1.1}^{p} \, a_{P.P}^{(n-p)} + a_{1.2}^{(p)} \, a_{P.P-1}^{(n-p)} + \ldots + a_{1.P}^{(p)} \, a_{P.1}^{(n-p)}.$$

On aura de même en général les deux équations

$$D_n = a_{1.\pi}^{(p)} \, a_{P.P-\pi+1}^{(n-p)} + a_{2.\pi}^{(p)} \, a_{P-1.P-\pi+1}^{(n-p)} + \ldots + a_{P.\pi}^{(p)} \, a_{1.P-\pi+1}^{(n-p)},$$

$$D_n = a_{\mu.1}^{(p)} \, a_{P-\mu+1.P}^{(n-p)} + a_{\mu.2}^{(p)} \, a_{P-\mu+1.P}^{(n-p)} + \ldots + a_{\mu.P}^{(p)} \, a_{P-\mu+1.1}^{(n-p)}.$$

Ces deux équations sont comprises dans la suivante

$$D_n = S^{P}\left(a_{\mu.\pi}^{(p)} \, a_{P-\mu+1.P-\pi+1}^{(n-p)}\right), \qquad \text{(XIV. 4.)}$$

qui a lieu également, soit que l'on considère le signe S comme relatif à l'indice μ, soit qu'on le considère comme relatif à l'indice π."

Taking as an illustration the case where $n=5$, $p=2$, and $\pi=7$ (that is, the ordinal number corresponding to the pair 2 5, of the suffixes 1, 2, 3, 4, 5), and translating literally from Cauchy's notation into our own, we have

$$|a_{11}a_{22}a_{33}a_{44}a_{55}| = |a_{12}a_{25}|\cdot|a_{31}a_{43}a_{54}| - |a_{12}a_{35}|\cdot|a_{21}a_{43}a_{54}| + \ldots$$
$$\ldots\ldots + |a_{42}a_{55}|\cdot|a_{11}a_{23}a_{34}|.$$

With the same certainty of touch and with still greater conciseness, all the identities directly obtainable by Bézout's *Méthode pour trouver des fonctions . . . . qui soient zéro par elles-mêmes*, are formulated as one general identity, and established on a proper basis. The paragraph is (p. 100)—

"D_n étant une fonction symétrique alternée des indices du système $(a_{1\cdot n})$ doit se réduire à zéro, lorsqu'on y remplace un de ces indices par un autre. Si l'on opère de semblables remplacemens à l'égard des indices qui occupent la première place dans le système $(a_{1\cdot n})$, et qui entrent dans la combinaison (μ), cette même combinaison se trouvera transformée en une autre que je désignerai par (ν), et $a^{(p)}_{\mu.\pi}$ sera changé en $a^{(p)}_{\nu.\pi}$. D'ailleurs, en supposant le signe S relatif à π, on a

$$D_n = S^P\left(a^{(p)}_{\mu.\pi}\, a^{(n-p)}_{P-\mu+1.P-\pi+1}\right);$$

on aura donc par suite

$$0 = S^P\left(a^{(p)}_{\nu.\pi}\, a^{(n-p)}_{P-\mu+1.P-\pi+1}\right). \quad (\text{XII. } 7;\ \text{XXIII. } 3).$$

On aurait de même, en supposant le signe S relatif à l'indice μ, et en désignant par (τ) une nouvelle combinaison différente de (π)

$$0 = S^P\left(a^{(p)}_{\mu.\tau}\, a^{(n-p)}_{P-\mu+1.P-\pi+1}\right)." \quad (\text{XII. } 7;\ \text{XXIII. } 3).$$

As this theorem is twin with the preceding, it is best to illustrate it by the same special case. By so doing, indeed, both theorems become more readily grasped and their details better understood. Taking then as before $n=5$, $p=2$ and $\pi=7$, we first form the determinants which Cauchy would have denoted by

$$a^{(2)}_{1.7}\ ,\ a^{(2)}_{2.7}\ ,\ \ldots\ldots\ ,\ a^{(2)}_{10.7}\ ,$$

and which we denote by

$$|a_{12}a_{25}|\ ,\ |a_{12}a_{35}|\ ,\ \ldots\ldots\ ,\ |a_{42}a_{55}|\,.$$

Next, for cofactors, we form the determinants which are complementary, not of these, as in the preceding theorem, but of the members of one of the nine other groups corresponding to the values 1, 2, 3, 4, 5, 6, 8, 9, 10 of π,—say the group

$$a^{(2)}_{1.6}\ ,\ a^{(2)}_{2.6}\ ,\ \ldots\ldots\ ,\ a^{(2)}_{10.6}\,.$$

These complementaries being

$$|a_{31}a_{43}a_{55}|\ ,\ |a_{21}a_{43}a_{55}|\ ,\ \ldots\ldots\ ,\ |a_{11}a_{23}a_{35}|\,,$$

we have the desired identity

$$0 = |a_{12}a_{25}|\cdot|a_{31}a_{43}a_{55}| - |a_{12}a_{35}|\cdot|a_{21}a_{43}a_{55}| \ldots\ldots + |a_{42}a_{55}|\cdot|a_{11}a_{23}a_{35}|\,,$$

the right-hand side of which is nothing more than an expansion of the zero determinant which arises from the determinant $|a_{11}a_{22}a_{33}a_{44}a_{55}|$ "lorsqu'on y remplace un des indices par un autre," viz., the second 4 by 5.

With the help of these two theorems a third theorem of almost equal importance is derived, viz., regarding the product of the determinants of two complementary systems. Denoting the determinant of the system

$$\left(a_{1.\mathrm{P}}^{(p)}\right) \quad \text{by} \quad \mathrm{D}_{\mathrm{P}}^{(p)},$$

and that of the complementary system

$$\left(a_{1.\mathrm{P}}^{(n-p)}\right) \quad \text{by} \quad \mathrm{D}_{\mathrm{P}}^{(n-p)},$$

and multiplying the two determinants together, we see with Cauchy that by (XIV. 4) the principal "termes" of the resulting determinant are each equal to

$$\mathrm{D}_n,$$

and by (XII. 7) all the other "termes" are equal to zero. Consequently

$$\mathrm{D}_{\mathrm{P}}^{(p)} \cdot \mathrm{D}_{\mathrm{P}}^{(n-p)} = (\mathrm{D}_n)^{\mathrm{P}} \qquad \text{(XLII.)}$$

As an example of this theorem, it may be added that the product of the two determinants printed above (p. 114) to illustrate the notation

$$\left(a_{1.\mathrm{P}}^{(p)}\right),$$

that is to say, the determinants of the systems

$$\left(a_{1.10}^{(2)}\right), \left(a_{1.10}^{(3)}\right),$$

is equal to

$$|a_{1\cdot 1}a_{2\cdot 2}a_{3\cdot 3}a_{4\cdot 4}a_{5\cdot 5}|^{10}.$$

In connection with all the three theorems, the special case, $p=1$, is given, so that their relation to previously well-known theorems (VI., XII., XXI.) may be noted. It is also pointed out, that when in the third theorem n is even and $p=\frac{1}{2}n$, the result takes the interesting form

$$\mathrm{D}_{\mathrm{P}}^{\left(\frac{n}{2}\right)} = (\mathrm{D}_n)^{\frac{\mathrm{P}}{2}}, \qquad \text{(XLII. 2).}$$

This brings us to the last section of the memoir, the fourth, bearing the heading

Des Systèmes d'Equations dérivées et de leur Déterminans.

What it is concerned with is the relations subsisting between a "derived system" of the product-determinant

$$\begin{vmatrix} m_{1\cdot1} & m_{1\cdot2} & \dots & m_{1\cdot n} \\ m_{2\cdot1} & m_{2\cdot2} & \dots & m_{2\cdot n} \\ \cdot & \cdot & \cdot & \cdot \\ m_{n\cdot1} & m_{n\cdot2} & \dots & m_{n\cdot n} \end{vmatrix},$$

and the corresponding "derived systems" of the factors

$$\begin{vmatrix} a_{1\cdot1} & a_{1\cdot2} & \dots & a_{1\cdot n} \\ a_{2\cdot1} & a_{2\cdot2} & \dots & a_{2\cdot n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n\cdot1} & a_{n\cdot2} & \dots & a_{n\cdot n} \end{vmatrix}, \quad \begin{vmatrix} \alpha_{1\cdot1} & \alpha_{1\cdot2} & \dots & \alpha_{1\cdot n} \\ \alpha_{2\cdot1} & \alpha_{2\cdot2} & \dots & \alpha_{2\cdot n} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{n\cdot1} & \alpha_{n\cdot2} & \dots & \alpha_{n\cdot n} \end{vmatrix};$$

in other words, the relations which must connect the systems

$$\left(a^{(p)}_{1.\mathrm{P}}\right), \left(\alpha^{(p)}_{1.\mathrm{P}}\right), \left(m^{(p)}_{1.\mathrm{P}}\right)$$

by reason of the relations

$$\Sigma[\mathrm{S}^n(a_{\nu\cdot1}\,\alpha_{\mu\cdot1}) = m_{\mu.\nu}]$$

(given in full above on p. 107) which connect the systems

$$(a_{1\cdot n}), \ (\alpha_{1\cdot n}), \ (m_{1\cdot n}).$$

First of all, attention is concentrated on a single "terme" of the system

$$\left(m^{(p)}_{1.\mathrm{P}}\right),$$

or, as we should nowadays say, on a minor of the product-determinant. The process of reasoning, which occupies about four quarto pages, is exactly analogous to that previously followed in dealing with the product-determinant itself; and the result obtained is

$$m^{(p)}_{\mu.\nu} = \mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\nu.1}\,\alpha^{(p)}_{\mu.1}\right), \qquad \text{(XVIII. 5)}$$

where S^{P} is meant to indicate that the terms on the right-hand side are got by changing the second suffixes into 2, 3, 4, . . . , P in succession. Speaking roughly and in modern phraseology, we may say that this means that

Any minor of a product-determinant is expressible as a sum of products of minors of the two factors. (XVIII. 5.)

Cauchy then proceeds (p. 107)—

"Si dans cette équation [XVIII. 5] on donne successivement à μ et à ν toutes les valeurs entières depuis 1 jusqu'à P, on aura un système d'équations symétriques de l'ordre P, que l'on pourra représenter par le symbole

$$(63) \qquad \Sigma\left\{ S^{P}\left(a_{\nu.1}^{(p)}\, a_{\mu.1}^{(p)}\right) = m_{\mu.\nu}^{(p)} \right\},$$

P étant toujours égal à

$$\frac{n(n-1)\; .\; .\; .\; .\; .\; (n-p+1)}{1\,.\,2\,.\,3\;.\;.\;.\;p}.$$

Pour déduire des équations

$$\Sigma\,[S^{n}(a_{\nu\cdot 1}\; a_{\mu\cdot 1}) = m_{\mu\cdot\nu}]$$

les équations (63), il suffit évidemment de remplacer les trois systèmes de quantités

$$(a_{1\cdot n}) \quad , \quad (a_{1\cdot n}) \quad , \quad (m_{1\cdot n})$$

par les systèmes dérivés de même ordre

$$\left(a_{1\cdot P}^{(p)}\right) , \left(a_{1.P}^{(p)}\right) , \left(m_{1.P}^{(p)}\right).$$

Je dirai pour cette raison que le second système d'équations est dérivé du premier." (XLI. 6.)

The close outward resemblance here noted between the original and the derived system of connecting equations is due of course to the choice of the notation

$$a_{1.P}^{(p)}$$

for the minors of the determinant

$$S \pm (a_{1\cdot 1}a_{2\cdot 2}\ldots a_{n\cdot n}),$$

and is so far a recommendation of that notation.

From the system of equations (63) two deductions follow immediately. In regard to the first Cauchy's words are (p. 108)—

"Désignons par

$$\delta_{P}^{(p)} \quad , \quad D_{P}^{(p)} \quad , \quad M_{P}^{(p)}$$

les déterminans des trois systémes

$$\left(a_{1.P}^{(p)}\right),\ \left(a_{1.P}^{(p)}\right),\ \left(m_{1.P}^{(p)}\right);$$

on aura en vertu des équations (63)

$$(65) \qquad M_P^{(p)} = D_P^{(p)} \delta_P^{(p)}.\text{"} \qquad (\text{XLIII.})$$

The enunciation of this in modern phraseology would be—

Any compound of a product-determinant is equal to the product of the corresponding compounds of the two factors. (XLIII.)

The next deduction is stated equally succinctly (p. 109)—

"Si l'on ajoute entre elles les équations (63) on aura la suivante,

$$(66) \qquad S^P \left\{ S^P\left(a_{\mu.\nu}^{(p)}\right) S^P\left(a_{\mu.\nu}^{(p)}\right) \right\} = S^P S^P\left(m_{\mu.\nu}^{(p)}\right), \qquad (\text{XXX. } 2)$$

le premier signe S, c'est-à-dire le signe extérieur, étant relatif à l'indice ν, et les autres, c'est-à-dire les signes intérieurs, étant relatifs à l'indice μ."

This (66) corresponds to (XL.) as (65) corresponds to the multiplication theorem

$$M_n = D_n \delta_n,$$

the transition from the general to the particular being effected in both cases by putting $p = 1$.

With these deductions, the 4th Section practically comes to an end; but one or two results, intentionally omitted in the account of the 2nd Section because they seemed to belong naturally to the 4th, fall now to be noted.

The first is very simple. It arises (p. 91) from observing that

$$\begin{aligned} (D_n)^{n-1} \times (\delta_n)^{n-1} &= (D_n\delta_n)^{n-1}, \\ \text{and } \therefore \qquad &= (M_n)^{n-1} \end{aligned}$$

by the multiplication-theorem. The result (XXI. 2) above (p. 110), is then thrice applied, and a theorem at once takes shape, which in later times we find enunciated as follows:—

The adjugate of the product-determinant is equal to the product of the adjugates of the two factors. (XLIII. 2.)

It is not noted, however, by Cauchy that this is but a case of XLIII., viz., where $p = n - 1$.

The next is

$$\Sigma\left[S^n(m_{1\cdot\nu}\ b_{1\cdot\mu}) = D_n a_{\nu\cdot\mu}\right],$$

or $$\Sigma\left[S^n(m_{\mu\cdot 1}\ \beta_{1\cdot\nu}) = \delta_n a_{\mu.\nu}\right] \qquad (\text{XLIV.})$$

It is nothing more than the result of solving the $n.n$ equations

$$(33) \quad \Sigma\left[S^n(a_{\nu\cdot 1}\ a_{\mu\cdot 1}) = m_{\mu\cdot\nu}\right]$$

first, in columns, for all the a's, and secondly, in rows for all the a's.

The last is

$$\Sigma\left[S^n(a_{1\cdot\mu}\ r_{\nu\cdot 1}) = \delta_n b_{\nu\cdot\mu}\right]$$

or $$\Sigma\left[S^n(a_{1\cdot\nu}\ r_{1\cdot\mu}) = D_n\beta_{\mu\cdot\nu}\right] \qquad (\text{XLIV. } 2)$$

where $(r_{1\cdot n})$ is the system adjugate to $(m_{1\cdot n}.)$. It is obtained from the $n.n$ equations (XLIV.) just as they were obtained from the $n.n$ equations (33), use being made of the theorem

$$M_n = D_n\delta_n.$$

In concluding, Cauchy refers to Binet's researches on similar matters. Most of what he says in regard to them has already been given (see p. 92 above). The rest of it is as follows (p. 111):—

> "Il [Binet] me dit en outre qu'il avait généralisé le théorème dont il s'agit [$M_n = D_n\delta_n$], en substituant au produit de deux résultantes des sommes de produits de même espèce. J'avais dès lors déjà démontré le théorème suivant :
>
> *D'un système quelconque d'équations symétriques on peut déduire cinq autres systèmes du même ordre ; mais on n'en saurait déduire un plus grand nombre.*
>
> J'ai démontré depuis à l'aide des méthodes précédentes cet autre théorème :
>
> *D'un système quelconque d'équations symétriques de l'ordre* n, *on peut toujours déduire deux systèmes d'équations symétriques de l'ordre*
>
> $$\frac{n(n-1)}{2},$$
>
> *deux systèmes d'équations symétriques de l'ordre*
>
> $$\frac{n(n-1)(n-2)}{1.2.3}, \text{ \&c. } \ldots\ldots$$
>
> En ajoutant entre elles les équations symétriques comprises dans un même système, on obtient, comme on l'a vu, les

formules (50), (51) et (70) qui me paraissent devoir être semblables à celles dont M. Binet m'a parlé."

The last sentence here raises an important question for the historian to settle, viz., whether Cauchy is to share with Binet the credit of the generalisation of the multiplication-theorem. The identities on which the claim is based are—

$$S^n\{S^n(a_{\mu\cdot\nu})\ \ S^n(a_{\mu\cdot\nu})\} = S^nS^n(m_{\mu\cdot\nu}) \qquad (50)$$

$$S^n\{S^n(\beta_{\mu\cdot\nu})\ \ S^n(b_{\mu\cdot\nu})\} = S^nS^n(r_{\mu\cdot\nu}) \qquad (51)$$

$$S^P\left\{S^P\left(a^{(p)}_{\mu.\nu}\right)S^P\left(a^{(p)}_{\mu.\nu}\right)\right\} = S^PS^P\left(m^{(p)}_{\mu.\nu}\right) \qquad (70)$$

The first of these, given formerly (p. 110) in the uncontracted form

$$\begin{aligned} &(a_{1\cdot1}+a_{2\cdot1}+\ldots+a_{n\cdot1})\,(a_{1\cdot1}+a_{2\cdot1}+\ldots+a_{n\cdot1})\\ &+(a_{1\cdot2}+a_{2\cdot2}+\ldots+a_{n\cdot2})\,(a_{1\cdot2}+a_{2\cdot2}+\ldots+a_{n\cdot2})\\ &+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+(a_{1\cdot n}+a_{2\cdot n}+\ldots+a_{n\cdot n})\,(a_{1\cdot n}+a_{2\cdot n}+\ldots+a_{n\cdot n})\\ =\ &\quad m_{1\cdot1}+m_{2\cdot1}+\ldots+m_{n\cdot1}\\ &+m_{1\cdot2}+m_{2\cdot2}+\ldots+m_{n\cdot2}\\ &+\ldots\ldots\ldots\\ &+m_{1\cdot n}+m_{2\cdot n}+\ldots+m_{n\cdot n} \end{aligned}$$

where $\quad m_{\mu.\nu} = a_{\mu\cdot1}a_{\nu\cdot1}+a_{\mu\cdot2}a_{\nu\cdot2}+\ldots+a_{\mu\cdot n}a_{\nu\cdot n}\,,$

may be at once left out of consideration; it is not even a case of the multiplication-theorem. Cauchy, we may be sure, mentioned it only because it is the first of the series to which (51) and (70) belong. The next concerns the systems

$$(\beta_{1\cdot n})\ ,\ (b_{1\cdot n})\ ,\ (r_{1\cdot n})$$

adjugate to the systems

$$(a_{1\cdot n})\ ,\ (a_{1\cdot n})\ ,\ (m_{1\cdot n})$$

dealt with in (50). It indeed is comparable with Binet's theorem; but as it is only a case of (70),—the minors in (70) being of any order whatever, whereas in (51) they are the principal minors,—we may without loss pass it over. Directing our attention, then, to (70) let us for the sake of greater definiteness take the case where $n=5$ and $p=2$, and where consequently $P=\dfrac{5\,.\,4}{1\,.\,2}=10$. The theorem then becomes

$$S^{10}\left\{S^{10}\left(\alpha^{(2)}_{\mu.\nu}\right)S^{10}\left(a^{(2)}_{\mu.\nu}\right)\right\} = S^{10}S^{10}\left(m^{(2)}_{\mu.\nu}\right).$$

For the purpose of comparison with Binet's result, it is absolutely necessary, however, to depart from this exceedingly condensed mode of statement. Remembering that the inner S's refer always to the first suffix, and the outer to the second suffix, we obtain the more developed form

$$\begin{aligned}
&\left(\alpha^{(2)}_{1.1} + \alpha^{(2)}_{2.1} + \ldots + \alpha^{(2)}_{10.1}\right)\left(a^{(2)}_{1.1} + a^{(2)}_{2.1} + \ldots + a^{(2)}_{10.1}\right)\\
+&\left(\alpha^{(2)}_{1.2} + \alpha^{(2)}_{2.2} + \ldots + \alpha^{(2)}_{10.2}\right)\left(a^{(2)}_{1.2} + a^{(2)}_{2.2} + \ldots + a^{(2)}_{10.2}\right)\\
+&\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
+&\left(\alpha^{(2)}_{1.10} + \alpha^{(2)}_{2.10} + \ldots + \alpha^{(2)}_{10.10}\right)\left(a^{(2)}_{1.10} + a^{(2)}_{2.10} + \ldots + a^{(2)}_{10.10}\right)\\
=\ &\ m^{(2)}_{1.1} + m^{(2)}_{2.1} + \ldots + m^{(2)}_{10.1}\\
+&\ m^{(2)}_{1.2} + m^{(2)}_{2.2} + \ldots + m^{(2)}_{10.2}\\
+&\ \ldots\ldots\\
+&\ m^{(2)}_{1.10} + m^{(2)}_{2.10} + \ldots + m^{(2)}_{10.10}\,.
\end{aligned}$$

Interpreting now the suffixes and superfixes of the α's, a's, and m's, after the manner already described,—any suffix r signifying along with the superfix (2) the r^{th} combination of two numbers taken from 1, 2, 3, 4, 5, we finally reach the suitable form

$$\begin{aligned}
&\left\{|\alpha_{1\cdot1}\alpha_{2\cdot2}| + |\alpha_{1\cdot1}\alpha_{3\cdot2}| + \ldots\ldots + |\alpha_{3\cdot1}\alpha_{4\cdot2}| + |\alpha_{3\cdot1}\alpha_{5\cdot2}| + |\alpha_{4\cdot1}\alpha_{5\cdot2}|\right\}\\
&\left\{|a_{1\cdot1}a_{2\cdot2}| + |a_{1\cdot1}a_{3\cdot2}| + \ldots\ldots + |a_{3\cdot1}a_{4\cdot2}| + |a_{3\cdot1}a_{5\cdot2}| + |a_{4\cdot1}a_{5\cdot2}|\right\}\\
+&\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
+&\left\{|\alpha_{1\cdot3}\alpha_{2\cdot5}| + |\alpha_{1\cdot3}\alpha_{3\cdot5}| + \ldots\ldots + |\alpha_{3\cdot3}\alpha_{4\cdot5}| + |\alpha_{3\cdot3}\alpha_{5\cdot5}| + |\alpha_{4\cdot3}\alpha_{5\cdot5}|\right\}\\
&\left\{|a_{1\cdot3}a_{2\cdot5}| + |a_{1\cdot3}a_{3\cdot5}| + \ldots\ldots + |a_{3\cdot3}a_{4\cdot5}| + |a_{3\cdot3}a_{5\cdot5}| + |a_{4\cdot3}a_{5\cdot5}|\right\}\\
+&\left\{|\alpha_{1\cdot4}\alpha_{2\cdot5}| + |\alpha_{1\cdot4}\alpha_{3\cdot5}| + \ldots\ldots + |\alpha_{3\cdot4}\alpha_{4\cdot5}| + |\alpha_{3\cdot4}\alpha_{5\cdot5}| + |\alpha_{4\cdot4}\alpha_{5\cdot5}|\right\}\\
&\left\{|a_{1\cdot4}a_{2\cdot5}| + |a_{1\cdot4}a_{3\cdot5}| + \ldots\ldots + |a_{3\cdot4}a_{4\cdot5}| + |a_{3\cdot4}a_{5\cdot5}| + |a_{4\cdot4}a_{5\cdot5}|\right\}\\
=\ &\ |m_{1\cdot1}m_{2\cdot2}| + |m_{1\cdot1}m_{3\cdot2}| + \ldots\ldots + |m_{3\cdot1}m_{4\cdot2}| + |m_{3\cdot1}m_{5\cdot2}| + |m_{4\cdot1}m_{5\cdot2}|\\
+&\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
+&\ |m_{1\cdot3}m_{2\cdot5}| + |m_{1\cdot3}m_{3\cdot5}| + \ldots\ldots + |m_{3\cdot3}m_{4\cdot5}| + |m_{3\cdot3}m_{5\cdot5}| + |m_{4\cdot3}m_{5\cdot5}|\\
+&\ |m_{1\cdot4}m_{2\cdot5}| + |m_{1\cdot4}m_{3\cdot5}| + \ldots\ldots + |m_{3\cdot4}m_{4\cdot5}| + |m_{3\cdot4}m_{5\cdot5}| + |m_{4\cdot4}m_{4\cdot5}|,
\end{aligned}$$

where $$m_{\mu\cdot\nu} = \alpha_{\mu\cdot1}a_{\nu\cdot1} + \alpha_{\mu\cdot2}a_{\nu\cdot2} + \ldots + \alpha_{\mu\cdot5}a_{\nu\cdot5}\,.$$

The series of suffixes for the α's, a's, and m's are seen to be the same,

the series of pairs of first suffixes in every row and the series of pairs of second suffixes in every column being

12, 13, 14, 15, 23, 24, 25, 34, 35, 45;

that is to say, the combinations arranged in ascending order, of the numbers 1, 2, 3, 4, 5, taken two at a time. On the first side of the identity are 10 products, and as both factors of each product contain 10 terms, the result of the multiplification would be to produce 1000 terms of the form

$$|a_{rp}a_{sq}|.\ a_{mp}a_{nq}| ,$$

the whole expansion in fact being

$$\sum_{\substack{q=2\\p<q}}^{q=5}\ \sum_{\substack{s=2\\r<s}}^{s=5}\ \sum_{\substack{n=2\\m<n}}^{n=5} |a_{rp}a_{sq}|.|a_{mp}a_{nq}| .$$

On the right hand side are 100 terms of the form

$$|m_{rp}m_{sq}| ,$$

and if a proof of the identity were wanted, we should only have to show that each of the 100 terms of the latter kind gives rise to a particular 10 terms of the former kind. This, too, it is interesting to note, Cauchy himself could have done. For example, the last of the 100 terms,

$$|m_{44}m_{55}|$$

$$= \begin{vmatrix} a_{41}a_{41}+a_{42}a_{42}+\ldots+a_{45}a_{45} & a_{41}a_{51}+a_{42}a_{52}+\ldots+a_{45}a_{55} \\ a_{51}a_{41}+a_{52}a_{42}+\ldots+a_{55}a_{45} & a_{51}a_{51}+a_{52}a_{52}+\ldots+a_{55}a_{55} \end{vmatrix},$$

$$= \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \times \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

$$= \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} . \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} + \begin{vmatrix} a_{41} & a_{43} \\ a_{51} & a_{53} \end{vmatrix} . \begin{vmatrix} a_{41} & a_{43} \\ a_{51} & a_{53} \end{vmatrix} + \ldots\ldots + \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} . \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix},$$

which is nothing more than Cauchy's formula (62)

$$m^{(p)}_{\mu.\nu} = \mathrm{S}^{\mathrm{P}}\left(a^{(p)}_{\nu.1}\, a^{(p)}_{\mu.1}\right),$$

when we put $\mu = 10 = \nu$, and $p = 2$. Instead of 1000 terms on the left-hand side and 100 on the right, we should clearly have for the general theorem P^3 terms on the left and P^2 terms on the right, P be it remembered being the combinatorial

$$\frac{n(n-1)(n-2)\ .\ .\ .\ (n-p+1)}{1\,.\,2\,.\,3\ .\ .\ .\ p}.$$

Leaving Cauchy, let us now return to Binet, and in order that the comparison between the two may be complete, let us formally enunciate in all its generality the latter's theorem also. Binet himself did not do this. After dealing with the case in which the determinants involved are of the 2nd order, he merely added (p. 289)—

"On aura encore pour les intégrales

$$\Sigma\{S(x,y',z'')\ S(\xi,v',\zeta'')\},\quad \Sigma\{S(t,x',y'',z''')\ S(\tau,\xi',v'',\zeta''')\},\ \&c.$$

des résultats semblables, savoir,

$$\Sigma\{S(x,y',z'')\ S(\xi,v',\zeta'')\}$$

$$= S_1\begin{cases} \Sigma x\xi\Sigma yv\Sigma z\zeta + \Sigma y\xi\Sigma zv\Sigma x\zeta + \Sigma z\xi\Sigma xv\Sigma y\zeta \\ -\Sigma x\xi\Sigma zv\Sigma y\zeta - \Sigma y\xi\Sigma xv\Sigma z\zeta - \Sigma z\xi\Sigma yv\Sigma x\zeta, \end{cases}$$

$$\Sigma\{S(t,x',y'',z''')\ S(\tau,\xi',v'',\zeta''')\}$$

$$= S_1\{\Sigma t\tau\Sigma x\xi\Sigma yv\Sigma z\xi + \Sigma x\tau\Sigma y\xi\Sigma tv\Sigma z\zeta + \&c.\}$$

&c."

With the help of modern phraseology, the general theorem thus intended to be indicated can be made sufficiently clear. Binet in effect says:—

Take s rectangular arrays each with m elements in the row and n elements in the column, m being greater than n, viz.—

$$\begin{array}{llll}
(a_1)_{11}(a_1)_{12}\cdots(a_1)_{1m} & (a_1)_{21}(a_1)_{22}\cdots(a_1)_{2m} & \cdot\ \cdot & (a_1)_{s1}(a_1)_{s2}\cdots(a_1)_{sm} \\
(a_2)_{11}(a_2)_{12}\cdots(a_2)_{1m} & (a_2)_{21}(a_2)_{22}\cdots(a_2)_{2m} & \cdot\ \cdot & (a_2)_{s1}(a_2)_{s2}\cdots(a_1)_{sm} \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
(a_n)_{11}(a_n)_{12}\cdots(a_n)_{1m}, & (a_n)_{21}(a_n)_{22}\cdots(a_n)_{2m}, & \cdot\ \cdot & (a_n)_{s1}(a_n)_{s2}\cdots(a_n)_{sm}
\end{array}$$

and other s rectangular arrays of the same kind, viz.—

$$\begin{array}{llll}
(b_1)_{11}(b_1)_{12}\cdots(b_1)_{1m} & (b_1)_{21}(b_1)_{22}\cdots(b_1)_{2m} & \cdot\ \cdot & (b_1)_{s1}(b_1)_{s2}\cdots(b_1)_{sm} \\
(b_2)_{11}(b_2)_{12}\cdots(b_2)_{1m} & (b_2)_{21}(b_2)_{22}\cdots(b_2)_{2m} & \cdot\ \cdot & (b_2)_{s1}(b_2)_{s2}\cdots(b_2)_{sm} \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
(b_n)_{11}(b_n)_{12}\cdots(b_n)_{1m} & (b_n)_{21}(b_n)_{22}\cdots(b_n)_{2m} & \cdot\ \cdot & (b_n)_{s1}(b_n)_{s2}\cdots(b_n)_{sm}
\end{array}$$

From each array, by taking every set of n columns, form $C_{m,n}$ determinants, arranging them in any order, provided it be the same for all the arrays. Add together all the 1st determinants formed from the first s arrays, and multiply the sum by the corresponding sum for the second s arrays; obtain the like product involving all the 2nd determinants, the like product involving all the 3rd determinants, and so on. Then, the sum of these products is equal to the sum of the products obtained by multiplying each array of the first set by each array of the second set.

Or we may put it alternatively as a formal proposition, thus:—

If s *rectangular arrays be taken, each with* m *elements in the row and* n *elements in the column,* m *being greater than* n, viz.

$$X_1,\ X_2,\ \dots\dots,\ X_s$$

and other s *rectangular arrays of the same kind,* viz.,

$$\Xi_1,\ \Xi_2,\ \dots\dots,\ \Xi_s;$$

and if the minor determinants of the nth *order formed from* X_1, X_2, ... , Ξ_{s-1}, Ξ_s *be*

$$\begin{array}{llll} x_{11} & x_{12} & \dots & x_{1C} \\ x_{21} & x_{22} & \dots & x_{2C} \\ \cdot & \cdot & \cdot & \cdot \\ x_{s1} & x_{s2} & \dots & x_{sC} \end{array} \qquad \begin{array}{llll} \xi_{11} & \xi_{12} & \dots & \xi_{1C} \\ \xi_{21} & \xi_{22} & \dots & \xi_{2C} \\ \cdot & \cdot & \cdot & \cdot \\ \xi_{s1} & \xi_{s} & \dots & \xi_{sC} \end{array}$$

then

$$\begin{aligned} &(x_{11}+x_{21}+\dots+x_{s1})(\xi_{11}+\xi_{21}+\dots+\xi_{s1}) \\ &+(x_{12}+x_{22}+\dots+x_{s2})(\xi_{12}+\xi_{22}+\dots+\xi_{s2}) \\ &+\dots\dots\dots\dots\dots\dots \\ &+(x_{1C}+x_{2C}+\dots+x_{sC})(\xi_{1C}+\xi_{2C}\dots+\xi_{sC}) \\ =\ &(X_1+X_2+\dots+X_s)(\Xi_1+\Xi_2+\dots+\Xi_s) \end{aligned}$$

where C stands for $C_{m,n}$ *i.e.*, $m(m-1.\ \dots(m-n+1)/1.2.3\dots.n.$

Now, counting the terms here as we did in the case of Cauchy's theorem, we see that on the left-hand side there are C multiplications to be performed, each giving rise to $s \times s$ terms, and that therefore the full number of terms in the development of this side is

$$s^2C;$$

also that on the right-hand side the number is

$$s^2.$$

whose numbers are II., III., IV., XII., XIV.; and we likewise see that information regarding them all will be got at p. 53 of the History. Speaking generally, more importance ought to be attached to the existence of numbers at the corner of a gnomon than elsewhere, because these indicate fresh departures in the theory. Sometimes, however, a fresh departure may have been very trivial, the real advance being indicated by a number well removed from the corner of a subsequent gnomon. Thus if we examine the history of the multiplication-theorem (Nos. XVII., XVIII.), we find the first step in the direction of it credited by the table to Lagrange, and subsequent steps to Gauss, Binet, and Cauchy; whereas careful investigation at the pages mentioned shows that what Lagrange accomplished was of exceedingly little moment, in comparison with the magnificent generalisation of Binet and Cauchy. Again, it must be borne in mind that all the results numbered in Roman figures are not of equal importance, it being well known that one theorem in any mathematical subject may have vastly more influence on the after development of the subject than half a dozen others. Such imperfections, however, being allowed for, the table will be found to afford a very ready means of estimating with considerable accuracy the proportionate importance to be assigned to the various early investigators of the theory.

If we look for a moment, in conclusion, at the nationality of the authors, one outstanding fact immediately arrests attention, viz., that almost every important advance is due to the mathematicians of France. Were the contributions of Bézout, Vandermonde, Laplace, Lagrange, Monge, Binet, and Cauchy left out, there would be exceedingly little left to any one else, and even that little would be of minor interest.

GERGONNE (1813).

[Développement de la théorie donnée par M. Laplace pour l'élimination au premier degré. *Annales de Mathématiques*, iv. pp. 148–155.]

This is such an exposition of the primary elements of the theory of determinants and their application to the solution of a set of simultaneous linear equations as might be given in the course of an

Rothe, 1800.	Gauss, 1801.	Monge, 1809.	Hirsch.	Binet, 1811.	Prasse.	Wronski, 1812.	Binet.	Cauchy.
					75			100
57, 57, 58, 59, 61					75, 77, 77,			101, 101
								103, 103
						78		99
								99
								102
								96
								102
								96, 103
								117
	64					78		
	67						80	109
				70, 72			80	120
								109
	66							
								118
		68						
	65							
		68						
				71				
							85	122
							86, 91	
							87, 91	
							88	
							88	
							90, 91	
							91	
								104
								106
								110
								110
								111, 114, 115, 115, 115, 121
								119, 119
								121, 122
								122, 123

whose numbers are II., III., IV., XII., XIV.; and we likewise see that information regarding them all will be got at p. 53 of the History. Speaking generally, more importance ought to be attached to the existence of numbers at the corner of a gnomon than elsewhere, because these indicate fresh departures in the theory. Sometimes, however, a fresh departure may have been very trivial, the real advance being indicated by a number well removed from the corner of a subsequent gnomon. Thus if we examine the history of the multiplication-theorem (Nos. XVII., XVIII.), we find the first step in the direction of it credited by the table to Lagrange, and subsequent steps to Gauss, Binet, and Cauchy; whereas careful investigation at the pages mentioned shows that what Lagrange accomplished was of exceedingly little moment, in comparison with the magnificent generalisation of Binet and Cauchy. Again, it must be borne in mind that all the results numbered in Roman figures are not of equal importance, it being well known that one theorem in any mathematical subject may have vastly more influence on the after development of the subject than half a dozen others. Such imperfections, however, being allowed for, the table will be found to afford a very ready means of estimating with considerable accuracy the proportionate importance to be assigned to the various early investigators of the theory.

If we look for a moment, in conclusion, at the nationality of the authors, one outstanding fact immediately arrests attention, viz., that almost every important advance is due to the mathematicians of France. Were the contributions of Bézout, Vandermonde, Laplace, Lagrange, Monge, Binet, and Cauchy left out, there would be exceedingly little left to any one else, and even that little would be of minor interest.

GERGONNE (1813).

[Développement de la théorie donnée par M. Laplace pour l'élimination au premier degré. *Annales de Mathématiques*, iv. pp. 148–155.]

This is such an exposition of the primary elements of the theory of determinants and their application to the solution of a set of simultaneous linear equations as might be given in the course of an

Theory of Determinants from 1693 *to* 1812.

Rothe, 1800.	Gauss, 1801.	Monge, 1809.	Hirsch.	Binet, 1811.	Prasse.	Wronski, 1812.	Binet.	Cauchy.
					75			100
57, 57, 58, 59, 61					75, 77, 77,			101, 101
								103, 103
						78		99
								99
								102
								96
								102
								96, 103
								117
	64					78		
	67						80	109
				70, 72			80	120
								109
	66							
		68						118
	65							
		68						
				71				
							85	122
							86, 91	
							87, 91	
							88	
							88	
							90, 91	
							91	
								104
								106
								110
								110
								111, 114, 115, 115, 115, 121
								119, 119
								121, 122
								122, 123

hour's lecture. It is confessedly founded on Laplace's memoir of 1772; but, though the matter of it is thus not original, it is nevertheless noteworthy on account of its brevity, clearness, and elegance.

The word "inversion" is introduced to denote (III. 20) what Cramer called a "dérangement," and then by easy steps the reader is led up to the theorem regarding the interchange of two non-contiguous letters.

> "(9) Donc, si l'on permute entre elles deux lettres non consécutives, on changera nécessairement l'espèce du nombre des inversions. Soit en effet n le nombre des lettres intermédiaires à ces deux-là; on pourra d'abord porter la lettre la plus à gauche immédiatement à gauche de l'autre, ce qui lui fera parcourir n places; puis remettre cette dernière à la place de la première; et, comme elle sera obligée de passer par-dessus celle-ci, elle se trouvera avoir parcouru $n+1$ places. Le nombre total des places parcourues par les deux lettres sera donc $2n+1$, et conséquemment l'espèce du nombre des inversions se trouvera changée." (III. 21)

This, it must be noted, is not identical with Rothe's proposition on the same subject, Gergonne's n being different from Rothe's d.

The proof, that a determinant vanishes if two of the letters bearing suffixes be the same, proceeds on the same lines as Rothe's, but is put very shortly and not less convincingly as follows:—

> "Supposons, en effet, que l'on change h en g, sans toucher à g ni aux indices. Soient, pour un terme pris au hasard dans le polynôme, p et q les indices respectifs de g et h; ce polynôme, renfermant toutes les permutations, doit avoir un autre terme ne différant uniquement de celui-la qu'en ce que c'est h qui y porte l'indice p et g l'indice q; et de plus (9) ces deux termes doivent être affectés de signes contraires; ils se détruiront donc, lorsqu'on changera h en g; et il en sera de même de tous les autres termes pris deux à deux." (XII. 8).

On putting "le polynôme D," *i.e.* the determinant $|a_1b_2c_3,\ldots|$ in the form

$$A_1a_1 + A_2a_2 + A_3a_3 + \ldots\ldots + A_ma_m,$$

this theorem of course leads at once to the identities

$$\left.\begin{array}{l} A_1 b_1 + A_2 b_2 + A_3 b_3 + \ldots + A_m b_m = 0 \\ A_1 c_1 + A_2 c_2 + A_3 c_3 + \ldots + A_m c_m = 0 \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\},$$

and these to the solution of m linear equations in m unknowns.

WRONSKI (1815).

[Philosophie de la Technie Algorithmique. Première Section, contenant la loi suprême et universelle des Mathématiques. Par Hoëné Wronski, pp. 175–181, &c. Paris.]

Here as in the *Réfutation* of 1812 "combinatory sums" make their appearance, as being necessary for the expression of the "loi suprème." Wronski's point of view is unaltered toward them. He now, however, calls them

Schin functions, (xv. 5)

from the letter formerly introduced to denote them, et "pour ne pas introduire de noms nouveaux"! Two or three pages are occupied with the statement of the recurrent law of formation (Bézout, 1764).

DESNANOT, P. (1819).

[Complément de la Théorie des Équations du Premier Degré, contenant . . . . . . Par P. Desnanot, Censeur au Collége Royal de Nancy, ... Paris.]

As far as can be gathered, Desnanot was acquainted with the writings of very few of his predecessors in the investigation of determinants. The only one he himself mentions is Bézout, and the first part of his work is in direct continuation of a topic which the latter had begun. His book is a marvel of laboured detail. No expositor could take more pains with his reader, space being held of no moment if clearness had to be secured. As might be expected, therefore, all that is really worth preserving of his work is but a small fraction of the 264 pages which he occupies in exposition.

The first chapter bears the heading

Recherche des Relations qui ont lieu entre le dénominateur et les numérateurs des valeurs générales des inconnues dans chaque système d'équations du premier degré ;

and, after a reference to the impossibility of obtaining any result in the case of one equation with one unknown, proceeds as follows :—

Si l'on a les deux équations

$$ax + by = c, \quad a'x + b'y = c',$$

elles donnent

$$x = \frac{cb' - bc'}{ab' - ba'}, \quad y = \frac{ac' - ca'}{ab' - ba'};$$

nommant D le dénominateur commun, N et N′ les numérateurs des valeurs de x et de y, nous aurons

$$D = ab' - ba', \quad N = cb' - bc', \quad N' = ac' - ca'.$$

Multiplions N par a, N′ par b et ajoutons, nous trouverons

$$aN + bN' = c(ab' - ba') = cD;$$

donc

$$aN + bN' = cD$$

Nous aurions de même, en multipliant N par a' et N′ par b', cette autre équation

$$a'N + b'N' = c'D."$$

With this may be compared Bézout's *Méthode pour trouver des fonctions . . . . qui soient zéro par elles-mêmes* (see p. 50).

Exactly the same method is followed with the set of equations

$$\left.\begin{aligned} ax + by + cz &= d \\ a'x + b'y + c'z &= d' \\ a''x + b''y + c''z &= d'' \end{aligned}\right\}.$$

Here fifteen relations are obtained, only seven of which are viewed as necessary, viz.,

$$\left.\begin{aligned} (ab' - ba')N' + (ac' - ca')N'' &= (ad' - da')D \\ (ab'' - ba'')N' + (ac'' - ca'')N'' &= (ad'' - da'')D \end{aligned}\right\},$$

$$\left.\begin{aligned} (da' - ad')N + (db' - bd')N' + (dc' - cd')N'' &= 0 \\ (da'' - ad'')N + (db'' - bd'')N' + (dc'' - cd'')N'' &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} a\,N + b\,N' + c\,N'' &= d\,D \\ a'N + b'N' + c'N'' &= d'D \\ a''N + b''N' + c''N'' &= d''D \end{aligned}\right\}.$$

From a modern point of view there are but *two* which are really different, viz.,

$$|ab'|.|ac'd''| - |ac'|.|ab'd''| + |ad'|.|ab'c''| = 0$$

and
$$a|bc'd''| - b|ac'd''| + c|ab'd''| - d|ab'c''| = 0\,,$$

the twelve quantities concerned being

$$\begin{matrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d''\,. \end{matrix}$$

The former is obtainable from Bézout's identity

$$|ab'c''|.|de'f''| - |ab'd''|.|ce'f''| + |ac'd''|.|be'f''| - |bc'd''|.|ae'f''| = 0$$

by putting

$$f\,, f'\,, f'' = 0\,,\, 0\,,\, 1$$

and
$$e\,, e',\, e'' = a\,,\, a',\, a''.$$

The other, as is well known, comes from Vandermonde.

Before proceeding to the case of four unknowns, a notation is introduced in the following words (p. 6):—

"Soient a, b, c, d, f, g, h, etc. des lettres représentant des quantités quelconques; k, l, m, p, q, r, etc. des indices d'accens qui doivent être placés à la droite des lettres. Au lieu de mettre ces indices comme des exposans, plaçons-les au-dessus des lettres qu'ils doivent affecter, de manière que $\overset{k}{a}$ désigne a affecté du nombre k d'accens; que $\overset{k}{a}\overset{l}{b}$ indique le produit de $\overset{k}{a}$ par $\overset{l}{b}$; ainsi de suite. Représentons la quantité $\overset{k}{a}\overset{l}{b} - \overset{k}{b}\overset{l}{a}$ par $\begin{pmatrix} k & l \\ a & b \end{pmatrix}$ de sorte que nous ayons cette équation

$$\begin{pmatrix} k & l \\ a & b \end{pmatrix} = \overset{k}{a}\overset{l}{b} - \overset{k}{b}\overset{l}{a}\,."$$

This being settled, the similar quantities of higher orders are defined by the equations

$$\begin{pmatrix} k & l & m \\ a & b & c \end{pmatrix} = \overset{m}{c}\begin{pmatrix} k & l \\ a & b \end{pmatrix} - \overset{l}{c}\begin{pmatrix} k & m \\ a & b \end{pmatrix} + \overset{k}{c}\begin{pmatrix} l & m \\ a & b \end{pmatrix},$$

$$\begin{pmatrix} k & l & m & p \\ a & b & c & d \end{pmatrix} = \overset{p}{d}\begin{pmatrix} k & l & m \\ a & b & c \end{pmatrix} - \overset{m}{d}\begin{pmatrix} k & l & p \\ a & b & c \end{pmatrix} + \overset{l}{d}\begin{pmatrix} k & m & p \\ a & b & c \end{pmatrix} - \overset{k}{d}\begin{pmatrix} l & m & p \\ a & b & c \end{pmatrix},$$

&c. &c. &c.

It is thus seen that Desnanot's definition is exactly the same as

Vandermonde's, and his notation essentially the same as Laplace's. To this definition and the proof of the theorem regarding the effect of the interchange of two indices or two letters seven pages are devoted, and then a fresh step is taken. The exact words of the original (pp. 13, 14) must be given, as they distinctly foreshadow a great theorem of later times.

" 14. Si nous développons cette expression

$$\binom{k\,l}{a\,b}\binom{m\,p}{a\,b}-\binom{k\,p}{a\,b}\binom{m\,l}{a\,b}$$

le résultat sera

$$\binom{k\,m}{a\,b}\binom{l\,p}{a\,b};$$

donc nous avons cette équation

$$(A)\quad \binom{k\,l}{a\,b}\binom{m\,p}{a\,b}-\binom{k\,p}{a\,b}\binom{m\,l}{a\,b}=\binom{k\,m}{a\,b}\binom{l\,p}{a\,b}.$$

15. De cette formule je vais en déduire d'autres. Je dis que si j'introduis la lettre c dans les seconds facteurs de chaque terme et en même temps l'indice k, l'équation subsistera encore, et que j'aurai

$$(B)\quad \binom{k\,l}{a\,b}\binom{k\,m\,p}{a\,b\,c}-\binom{k\,p}{a\,b}\binom{k\,m\,l}{a\,b\,c}=\binom{k\,m}{a\,b}\binom{k\,l\,p}{a\,b\,c}.$$

L'égalité serait prouvée si en développant les deux membres, les quantités multipliées par la même lettre c, affectée d'indices égaux, étaient égales dans chaque membre ; or j'ai

$$\left.\begin{array}{l}
\overset{p}{c}\binom{k\,l}{a\,b}\binom{k\,m}{a\,b}\\
-\overset{m}{c}\left(\binom{k\,l}{a\,b}\binom{k\,p}{a\,b}-\binom{k\,p}{a\,b}\binom{k\,l}{a\,b}\right)\\
+\overset{k}{c}\left(\binom{k\,l}{a\,b}\binom{m\,p}{a\,b}-\binom{k\,p}{a\,b}\binom{m\,l}{a\,b}\right)\\
-\overset{l}{c}\binom{k\,p}{a\,b}\binom{k\,m}{a\,b}
\end{array}\right\}=\left\{\begin{array}{l}
+\overset{p}{c}\binom{k\,m}{a\,b}\binom{k\,l}{a\,b}\\
\\
+\overset{k}{c}\binom{k\,m}{a\,b}\binom{l\,p}{a\,b}\\
-\overset{l}{c}\binom{k\,m}{a\,b}\binom{k\,p}{a\,b}.
\end{array}\right.$$

Les quantités multipliées par $\overset{p}{c}$, $\overset{m}{c}$ et $\overset{l}{c}$ dans chaque membre sont égales entr'elles, c'est évident ; et la formule (A) rend les coefficiens de $\overset{k}{c}$ égaux ; donc puisque dans (B), il n'y a que des termes multipliés par $\overset{p}{c}$, $\overset{m}{c}$, $\overset{k}{c}$, et $\overset{l}{c}$, je conclus que l'équation (B) est exacte."

Having thus shown that if in each of the second factors of the identity

$$|a_1b_2||a_3b_4| - |a_1b_3||a_2b_4| + |a_1b_4||a_2b_3| = 0 \qquad \text{(A)},$$

a new letter c be added and the index 1 be prefixed, the sign of equality may still be retained, so that we have a new identity

$$a_1b_2||a_1b_3c_4| - |a_1b_3||a_1b_2c_4| + |a_1b_4||a_1b_2c_3| = 0 \qquad \text{(B)};$$

he then goes on to prove in the same fashion that the first factors of this derived identity may be treated in a similar way with impunity, viz., that they may be extended by the appending of the letter c with a new index 5, so that we have a further derived identity

$$|a_1b_2c_5||a_1b_3c_4| - |a_1b_3c_5||a_1b_2c_4| + |a_1b_4c_5||a_1b_2c_3| = 0 \qquad \text{(C)},$$

already known to us from Monge.

And this is not all, for the next paragraph shows that these two extensions may be repeated in order as often as we please, the opening of the paragraph being as follows (p. 15):—

"17. Généralisons et prouvons que si la formule

$$\begin{pmatrix} k\,l\,..\,q \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,m\,..\,p \\ a\,b\,..\,c \end{pmatrix} - \begin{pmatrix} k\,m\,..\,l \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,p\,..\,q \\ a\,b\,..\,c \end{pmatrix} = \begin{pmatrix} k\,l\,..\,p \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,m\,..\,q \\ a\,b\,..\,c \end{pmatrix}$$

est vraie dans le cas où il y aurait n lettres comprises dans chaque facteur, elle sera encore vraie en ajoutant une nouvelle lettre d dans les seconds facteurs de chaque terme avec l'indice l qui n'y entre pas; et qu'ensuite, si l'on ajoute la même lettre d dans les premiers facteurs de chaque terme avec un nouvel indice r, l'égalité ne sera pas troublée.

Il s'agit donc de démontrer que ces deux formules sont exactes :

$$\begin{pmatrix} k\,l\,..\,q \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,m\,..\,l\,p \\ a\,b\,..\,c\,d \end{pmatrix} - \begin{pmatrix} k\,m\,..\,l \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,p\,..\,l\,q \\ a\,b\,..\,c\,d \end{pmatrix} = \begin{pmatrix} k\,l\,..\,p \\ a\,b\,..\,c \end{pmatrix}\begin{pmatrix} k\,m\,..\,l\,q \\ a\,b\,..\,c\,d \end{pmatrix},$$

(XXIII. 4)

$$\begin{pmatrix} k\,l\,..\,q\,r \\ a\,b\,..\,c\,d \end{pmatrix}\begin{pmatrix} k\,m\,..\,l\,p \\ a\,b\,..\,c\,d \end{pmatrix} - \begin{pmatrix} k\,m\,..\,l\,r \\ a\,b\,..\,c\,d \end{pmatrix}\begin{pmatrix} k\,p\,..\,l\,q \\ a\,b\,..\,c\,d \end{pmatrix} = \begin{pmatrix} k\,l\,..\,p\,r \\ a\,b\,..\,c\,d \end{pmatrix}\begin{pmatrix} k\,m\,..\,l\,q \\ a\,b\,..\,c\,d \end{pmatrix}.$$

The line of proof is still the same, and may be shortly indicated by treating the case

$$\text{(D)} \quad |a_1b_2c_5||a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5| + |a_1b_2c_3||a_1b_2c_4d_5| = 0,$$

which comes immediately after (C), and is derived from it by extending the factors *in which* a_1b_2 *does not occur.* Since by definition

$$|a_1b_2c_3d_4| = d_4|a_1b_2c_3| - d_3|a_1b_2c_4| + d_2|a_1b_3c_4| - d_1|a_2b_3c_4|,$$

and $$|a_1b_2c_3d_5| = d_5|a_1b_2c_3| - d_3|a_1b_2c_5| + d_2|a_1b_3c_5| - d_1|a_2b_3c_5|,$$

it follows that

$$\begin{aligned}&|a_1b_2c_5||a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5| \\ &\quad = \left.\begin{aligned}&\left\{d_4|a_1b_2c_5| - d_5|a_1b_2c_4|\right\}|a_1b_2c_3| \\ &+ \left\{|a_1b_2c_5||a_1b_3c_4| - |a_1b_2c_4||a_1b_3c_5|\right\}d_2 \\ &- \left\{|a_1b_2c_5||a_2b_3c_4| - |a_1b_2c_4||a_2b_3c_5|\right\}d_1.\end{aligned}\right\}\end{aligned}$$

But the cofactor here of d_2 is by (C) equal to

$$-|a_1b_4c_5||a_1b_2c_3|;$$

and the cofactor of d_1

$$= |a_2b_1c_5||a_2b_3c_4| - |a_2b_1c_4||a_2b_3c_5|,$$

and therefore by (C)

$$\begin{aligned}&= -|a_2b_1c_3||a_2b_4c_5|, \\ &= \quad |a_1b_2c_3||a_2b_4c_5|.\end{aligned}$$

Making these substitutions, we have

$$\begin{aligned}&|a_1b_2c_5\,a_1b_2c_3d_4| - |a_1b_2c_4||a_1b_2c_3d_5 \\ &= -|a_1b_2c_3|\left\{d_5|a_1b_2c_4| - d_4|a_1b_2c_5| + d_2|a_1b_4c_5| - d_1|a_2b_4c_5|\right\} \\ &= -|a_1b_2c_3||a_1b_2c_4d_5|,\end{aligned}$$

as was to be shown.

The next three cases are

$$|a_1b_2c_5d_6||a_1b_2c_3d_4| - |a_1b_2c_4d_6||a_1b_2c_3d_5| + |a_1b_2c_3d_6||a_1b_2c_4d_5| = 0 \text{ (E)}$$
$$|a_1b_2c_3d_4||a_1b_2c_3d_5e_6| - |a_1b_2c_3d_5||a_1b_2c_3d_4e_6| + |a_1b_2c_3d_6||a_1b_2c_3d_4e_5| = 0 \text{ (F)}$$
$$|a_1b_2c_3d_4e_7||a_1b_2c_3d_5e_6| - |a_1b_2c_3d_5e_7||a_1b_2c_3d_4e_6| + |a_1b_2c_3d_6e_7||a_1b_2c_3d_4e_5| = 0 \text{ (G).}$$

When the factors of each product are of the same order, as in (C), (E), (G) the identity is, in modern phraseology, an "extensional" of (A); that is to say, there is a part common to every factor of the identity, *e.g.*, a_1 in (C), a_1b_2 in (E), $a_1b_2c_3$ in (G), and this common part being deleted, the result is simply the identity

(A). When the factors of each product are of different orders, as in (B), (D), (F), the identity is an "extensional" of something still simpler than (A), viz.,

$$a_1|a_2b_3| - a_2|a_1b_3| + a_3|a_1b_2| = 0\,.$$

In exactly the same manner and at quite as great length the identity

$$\binom{k\ l}{a\ f}\binom{k\ r}{a\ g} - \binom{k\ l}{a\ g}\binom{k\ r}{a\ f} = \binom{k\ l\ r}{a\ f\ g}\binom{k}{a}$$

—already known to us from Lagrange—is made the source of a numerous progeny. By putting figures for k, l, . . . and at the same time writing them as suffixes, these identities, original and derived, take the form

$$|a_1f_2||a_1g_6| - |a_1g_2||a_1f_6| = |a_1f_2g_6||a_1|\,, \qquad (A')$$
$$|a_1f_2||a_1b_2g_6| - |a_1g_2||a_1b_2f_6| = |a_1f_2g_6||a_1b_2|\,, \qquad (B')$$
$$|a_1b_2f_3||a_1b_2g_6| - |a_1b_2g_3||a_1b_2f_6| = |a_1b_2f_3g_6||a_1b_2|\,, \qquad (C')$$
$$|a_1b_2f_3||a_1b_2c_3g_6| - |a_1b_2g_3||a_1b_2c_3f_6| = |a_1b_2f_3g_6||a_1b_2c_3|\,, \qquad (D')$$
$$|a_1b_2c_3f_4||a_1b_2c_3g_6| - |a_1b_2c_3g_4||a_1b_2c_3f_6| = |a_1b_2c_3f_4g_6||a_1b_2c_3|\,, \qquad (E')$$
$$|a_1b_2c_3f_4||a_1b_2c_3d_4g_6| - |a_1b_2c_3g_4||a_1b_2c_3d_4f_6| = |a_1b_2c_3f_4g_6||a_1b_2c_3d_4|\,, \qquad (F')$$
$$|a_1b_2c_3d_4f_5||a_1b_2c_3d_4g_6| - |a_1b_2c_3d_4g_5||a_1b_2c_3d_4f_6| = |a_1b_2c_3d_4f_5g_6||a_1b_2c_3d_4|\,. \qquad (G')$$

Of these (C′), (E′), (G′) deserve to be noted, being along with the original (A′) extensionals of the manifest identity

$$f_2g_6 - g_2f_6 = |f_2g_6|\,. \qquad (\text{XXIII. } 5).$$

On the other hand (B′), (D′), (F′) are essentially the same as (B), (D), (F) already obtained—a fact which Desnanot overlooks.

As the source of a third series of results, obtained in still the same way, the identity

$$\binom{k\ l}{a\ h}\binom{k\ l}{f\ g} - \binom{k\ l}{a\ g}\binom{k\ l}{f\ h} = \binom{k\ l}{a\ f}\binom{k\ l}{h\ g} \quad . \quad . \quad (A'')$$

is next taken. In reality, however, this does not differ from the first identity so treated, viz.,

$$\binom{k\ l}{a\ b}\binom{m\ p}{a\ b} - \binom{k\ p}{a\ b}\binom{m\ l}{a\ b} = \binom{k\ m}{a\ b}\binom{l\ p}{a\ b} \quad . \quad . \quad (A).$$

In (A) the letters ab remain unchanged throughout, and the indices vary; while in (A″) the indices remain the same, and the letters

vary. As we should now say, the difference is a mere matter of rows and columns. The derived identities (B″), (C″), (D″), . . . are consequently found to be quite the same as (B), (C), (D), ...

The fourth and last source made use of is the well-known theorem regarding the aggregate of products whose first factors constitute what Cauchy would have called a "suite verticale," and whose second factors are the cofactors, in the determinant of the system, of another "suite verticale." Desnanot however, viewing the theorem from a different stand-point, enunciates it as follows (p. 26):—

> "*Si l'on a* n *lettres* ab .... cdf, *et qu'on les combine* n − 1 *à* n − 1, *on aura* n *arrangemens* ab .... cd, ab .... cf, ab .... df, ... , a .... cdf, b .... cdf; *qu'on applique dans chaque arrangement les* n − 1 *indices* kl....mp, *ce qui donnera ces quantités*

$$\begin{pmatrix} k & \ldots & m & p \\ a & b \ldots & c & d \end{pmatrix}, \begin{pmatrix} k & l \ldots & m & p \\ a & b \ldots & c & f \end{pmatrix}, \begin{pmatrix} k & l \ldots & m & p \\ a & b \ldots & d & f \end{pmatrix}, \ldots\ldots \begin{pmatrix} k & l \ldots & m & p \\ a & \ldots c & d & f \end{pmatrix}, \begin{pmatrix} k & l \ldots & m & p \\ b & \ldots c & d & f \end{pmatrix};$$

> *et qu'ensuite on les multiplie chacune par la lettre qui n'entre pas dans l'arrangement en l'affectant d'un même indice et donnant au produit le signe* plus *ou le signe* moins, *suivant que la lettre multiplicateur occupe un rang impair ou pair dans les* n *lettres, en partant de la droite, la somme des produits sera zéro.*" (XII. 9).

Before proceeding to deduce others from it, he gives a proof of it for the case

$$(\mathrm{B}''') \quad \overset{p}{f}\begin{pmatrix} k & l \ldots & m & p \\ a & b \ldots & c & d \end{pmatrix} - \overset{p}{d}\begin{pmatrix} k & l \ldots & m & p \\ a & b \ldots & c & f \end{pmatrix} + \overset{p}{c}\begin{pmatrix} k & l \ldots & m & p \\ a & b \ldots & d & f \end{pmatrix} - \cdot \;\cdot\;\cdot\;\cdot$$
$$\mp \overset{p}{b}\begin{pmatrix} k & l \ldots & m & p \\ a & \ldots c & d & f \end{pmatrix} \pm \overset{p}{a}\begin{pmatrix} k & l \ldots & m & p \\ b & \ldots c & d & f \end{pmatrix} = 0.$$

The method of proof is interesting, because it depends almost entirely on the definition which Desnanot follows Vandermonde in using. It will be readily understood by seeing it applied to the simple case

$$b_1|b_2c_3d_4| - b_2|b_1c_3d_4| + b_3|b_1c_2d_4| - b_4\,|b_1c_2d_3| = 0.$$

Expanding each of the determinants $|b_2c_3d_4|$, $|b_1c_3d_4|$, ... in terms of the b's and their cofactors, we have

$$
\begin{aligned}
& b_1|b_2c_3d_4| - b_2|b_1c_3d_4| + b_3|b_1c_2d_4| - b_4|b_1c_2d_3| \\
= . & \left.\begin{array}{l}
b_1\{ \qquad\qquad b_2|c_3d_4| - b_3|c_2d_4| + b_4|c_2d_3|\} \\
- b_2\{ b_1|c_3d_4| \qquad\qquad - b_3|c_1d_4| + b_4|c_1d_3|\} \\
+ b_3\{ b_1|c_2d_4| - b_2|c_1d_4| \qquad\qquad + b_4|c_1d_2|\} \\
- b_4\{ b_1|c_2d_3| + b_2|c_1d_3| + b_3|c_1d_2| \qquad\qquad \}
\end{array}\right\}, \\
= 0,
\end{aligned}
$$

for the terms in the expanded form destroy each other in pairs.

The derived identities are obtained exactly in the manner followed by Bézout in 1779 (see pp. 51, 52). The fundamental identity is taken, say in the form

$$
\begin{aligned}
f_5|a_1b_2c_3d_4e_5| - e_5|a_1b_2c_3d_4f_5| + d_5|a_1b_2c_3e_4f_5| - c_5|a_1b_2d_3e_4f_5| \\
+ b_5|a_1c_2d_3e_4f_5| - a_5|b_1c_2d_3e_4f_5| = 0,
\end{aligned}
$$

and another instance is put alongside of it, in which the same letters and suffixes are involved, say

$$
\begin{aligned}
f_1|a_1b_2c_3d_4e_5| - e_1|a_1b_2c_3d_4f_5| + d_1|a_1b_2c_3e_4f_5| - c_1|a_1b_2d_3e_4f_5| \\
+ b_1|a_1c_2d_3e_4f_5| - a_1|b_1c_2d_3e_4f_5| = 0.
\end{aligned}
$$

One of the constituent determinants, say the last, $|b_1c_2d_3e_4f_5|$ is then eliminated by equalisation of coefficients and subtraction, the result being

$$
\begin{aligned}
|a_1f_5| \cdot |a_1b_2c_3d_4e_5| - |a_1e_5||a_1b_2c_3d_4f_5| + |a_1d_5||a_1b_2c_3e_4f_5| \\
+ |a_1c_5||a_1b_2d_3e_4f_5| - |a_1b_5||a_1c_2d_3e_4f_5| = 0 \quad (C''')
\end{aligned}
$$

In the next place, two additional instances of this derived identity are taken along with it, the first differing from it in having a 2 instead of a 5 in all the first factors, and the second in having a 2 instead of a 1; viz.,

$$
\begin{aligned}
|a_1f_2||a_1b_2c_3d_4e_5| - |a_1e_2||a_1b_2c_3d_4f_5| + |a_1d_2||a_1b_2c_3e_4f_5| \\
+ |a_1c_2||a_1b_2d_3e_4f_5| - |a_1b_2||a_1c_2d_3e_4f_5| = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
|a_2f_5||a_1b_2c_3d_4e_5| - |a_2e_5||a_1b_2c_3d_4f_5| + |a_2d_5||a_1b_2c_3e_4f_5| \\
+ |a_2c_5||a_1b_2d_3e_4f_5| - |a_2b_5||a_1c_2d_3e_4f_5| = 0.
\end{aligned}
$$

Multiplication by $b_2, -b_5, -b_1$ is then effected and addition performed, when by reason of such identities as

$$b_2|a_1f_5| - b_5|a_1f_2| - b_1|a_2f_5| = |a_1b_2f_5|,$$

and $$b_2|a_1b_5| - b_5|a_1b_2| - b_1|a_2b_5| = 0,$$

elimination of $|a_1c_2d_3e_4f_5|$ is produced, and the result takes the form

$$(\mathrm{D}''')\quad \begin{aligned} &|a_1b_2f_5||a_1b_2c_3d_4e_5| - |a_1b_2e_5||a_1b_2c_3d_4f_5| + |a_1b_2d_5||a_1b_2c_3e_4f_5| \\ &\qquad - |a_1b_2c_5||a_1b_2d_3e_4f_5| = 0. \end{aligned}$$

The process of derivation may be pursued further, giving next an identity in which the first factors are all of the fourth order. Desnanot says (pp. 31, 32)—

"Pour ne pas nous répéter constamment, nous dirons que cette formule s'étendrait à un nombre quelconque de lettres placées dans les premiers facteurs, et que

$$(\mathrm{H}''')\quad \begin{aligned} &\binom{k\,l\,\ldots\,p}{a\,b\,\ldots\,f}\binom{k\,l\,\ldots\,mp}{a\,b\,\ldots\,c\,d} - \binom{k\,l\,\ldots\,p}{a\,b\,\ldots\,d}\binom{k\,l\,\ldots\,mp}{a\,b\,\ldots\,c\,f} \\ &\qquad + \binom{k\,l\,\ldots\,p}{a\,b\,\ldots\,c}\binom{k\,l\,\ldots\,mp}{a\,b\,\ldots\,d\,f} - \;.\;\;.\;\;.\;\;.\;\;. = 0. \end{aligned}$$

Les termes sont alternativement positifs et négatifs, les indices sont les mêmes dans les premiers facteurs de chaque terme, ils font partie des indices qui se trouvent dans les autres facteurs et sont placés dans le même ordre; quant aux lettres, il y a ou une, ou deux, ou trois, etc. lettres communes aux seconds facteurs écrites toujours dans le même ordre et suivies de la $n^{\text{ième}}$ lettre qui n'entre pas dans les seconds facteurs; de sorte que s'il y a n' lettres communes à tous les facteurs, le nombre des termes de (H''') sera $n - n'$." (XXIII. 6)

The general result (H''') is simply what would now be called the extensional of the identity of Vandermonde from which Desnanot derives it.

Co-ordinate, in a sense, with the said identity, is that other which Desnanot uses as a definition; and this latter is the next of which the extensional is found. The process, so far as indicated, is exactly similar to that employed in the preceding case. The results obtained are

$$(\text{B}'''')\quad \begin{aligned}&\binom{p\,r}{a\,f}\binom{k\,l\ldots m\,p}{a\,b\ldots c\,d} - \binom{p\,r}{a\,d}\binom{k\,l\ldots m\,p}{a\,b\ldots c\,f} + \binom{p\,r}{a\,c}\binom{k\,l\ldots m\,p}{a\,b\ldots d\,f}\\ &\quad\ldots\ldots \mp \binom{p\,r}{a\,b}\binom{k\,l\ldots m\,p}{a\ldots c\,d\,f} = \binom{p}{a}\binom{k\,l\ldots m\,p\,r}{a\,b\ldots c\,d\,f}\end{aligned}$$

$$(\text{C}'''')\quad \begin{aligned}&\binom{k\,p\,r}{a\,b\,f}\binom{k\,l\ldots m\,p}{a\,b\ldots c\,d} - \binom{k\,p\,r}{a\,b\,d}\binom{k\,l\ldots m\,p}{a\,b\ldots c\,f}\\ &+\binom{k\,p\,r}{a\,b\,c}\binom{k\,l\ldots m\,p}{a\,b\ldots d\,f} - \ldots\ldots = \binom{k\,p}{a\,b}\binom{k\,l\ldots m\,p\,r}{a\,b\ldots c\,d\,f};\end{aligned}$$

and the general result including them is referred to. (XLVI.)

That they are extensionals of the definition is evident from the fact that the index p may be moved to the left so as to make $\overset{p}{a}$ common to every factor of (B''''), and $\overset{k}{a}\,\overset{p}{b}$ common to every factor of (C'''').

Still another series of results is obtained, but they are essentially the same as the foregoing, the difference again being merely a matter of rows and columns.

All these preparations having been made, Desnanot returns to the subject of the relations between the numerators and denominators of the values of the unknowns in a set of linear equations. Thirteen pages are occupied with the case of four unknowns, the number of relations found being 74, of which, after scrutiny, 14 are retained. The case of five unknowns, and the case of six unknowns are gone into with about as much detail, and then, lastly, the general set of n equations with n unknowns is dealt with. None of the relations obtained need be given, as they are all included in the identities which have been spoken of above as extensionals.

The second chapter (p. 94) bears the heading

Simplification des formules générales qui donnent les valeurs des inconnues dans les équations du premier degré, lorsqu'on veut les évaluer en nombres.

Here again the cases of three, four, five, six unknowns are dwelt upon with equal fulness in succession. The consideration of one of them will suffice to show the nature of the method, and will enable the reader to judge of the amount of labour saved by employing it. Choosing the case of four unknowns, we find at the outset the equations stated and the solution condensed as follows (p. 104) :—

"EQUATIONS DONNÉES.

$$\begin{aligned}
ax + by + cz + dt &= f\\
a'x + b'y + c'z + d't &= f'\\
a''x + b''y + c''z + d''t &= f''\\
a'''x + b'''y + c'''z + d'''t &= f'''.
\end{aligned}$$

CALCUL.

$$\begin{aligned}
ab' - ba' &= \alpha,\\
ab'' - b''a &= \beta,\\
a'b'' - b''a' &= \gamma,\\
ab''' - ba''' &= \delta,\\
a'b''' - b'a''' &= \epsilon;
\end{aligned}$$

$$\begin{aligned}
m &= c''\alpha - c'\beta + c\gamma,\\
n &= c'''\alpha - c'\delta + c\epsilon,\\
m' &= f''\alpha - f'\beta + f\gamma,\\
n' &= f'''\alpha - f'\delta + f\epsilon;
\end{aligned}$$

$$D = \frac{1}{\alpha}\left\{ m(ad''' - \delta d' + \epsilon d) - n(ad'' - \beta d' + \gamma\delta) \right\}.$$

$$N''' = \frac{1}{\alpha}\Big(mn' - nm' \Big),$$

$$N'' = \frac{1}{\alpha}\left\{ m'(ad''' - \delta d' + \epsilon d) - n'(ad'' - \beta d' + \gamma d) \right\},$$

$$\begin{aligned}
f\,D - cN'' - dN''' &= S,\\
f'D - c'N'' - d'N''' &= S',
\end{aligned}$$

$$N' = \frac{aS' - Sa'}{\alpha},$$

$$N = \frac{Sb' - bS'}{\alpha},$$

$$x = \frac{N}{D}, \quad y = \frac{N'}{D}, \quad z = \frac{N''}{D}, \quad t = \frac{N'''}{D}."$$

The explanation of the mode of procedure is not difficult to see.

(1) The determinants $|ab'|$, $|ab''|$, $|a'b''|$, $|ab'''|$, $|a'b'''$ are calculated.

(2) With the help of these are next got four of a higher order, viz. $|ab'c''|$, $|ab'c'''|$, $|ab'f''|$, $|ab'f'''|$.

(3) Two others of the same order, viz.

$$ad''' - \delta d' + \epsilon d, \quad ad'' - \beta d' + \gamma d$$

i.e.

$$ab'd'''|, \quad |ab'd''|,$$

having been calculated, the identity

$$|ab'| . \mathrm{D} = |ab'c''| . |ab'd'''| - |ab'c'''| . |ab'd''|$$

is used to find D.

(4) A similar identity

$$|ab'| . \mathrm{N}''' = |ab'c''| . |ab'f'''| - |ab'c'''| . |ab'f''|$$

is used to find N‴.

(5) A similar identity

$$| ab' | . \mathrm{N}'' = |ab'f''| . |ab'd'''| - |ab'f'''| . |ab'd''|$$

is used to find N″.

(6) Two subsidiary quantities S, S′ are calculated, the first being

$$= f\,|ab'c''d'''| - c|ab'f''d'''| - d|ab'c''f'''| ,$$

and the second

$$= f'\,|ab'c''d'''| - c'\,|ab'f''d'''| - d'|ab'c''f'''| .$$

(7) From these N′ and N are readily got. For evidently

$$a\mathrm{S}' - \mathrm{S}a'$$

$$= |af'| . |ab'c''d'''| - |ac'| . |ab'f''d'''| - |ad'| . |ab'c''f'''|$$

and this by a previous theorem

$$= |ab'| . |af'c''d'''| ,$$

$$= |ab'| . \mathrm{N}' . \qquad \text{(XIII. 3.)}$$

The third chapter consists of a lengthy examination (pp. 157–264) of the singular cases met with in the solution of linear equations, and does not at present concern us.

CAUCHY (1821).

[Cours d'Analyse de l'Ecole Royale Polytechnique I. xvi. + 576 pp. Paris.]

When Cauchy came to write his *Course of Analysis*, afterwards so well known, he did not fail to assign a position in it to the subject of his memoir of 1812. The third chapter bears the heading, "*Des Fonctions Symétriques et des Fonctions Alternées.*"

It occupies, however, only fifteen pages (pp. 70–84), and of these only nine are devoted to alternating functions and the solution of simultaneous linear equations. Of course, in so limited a space, the merest sketch of a theory is all that is possible. An alternating function is first defined, the word "alternée" being now set in contrast with "symétrique," and not, as formerly, with "permanente." Functions other than those that are rational and integral being left aside, the latter, if alternating, are shown (1) to consist of as many positive as negative terms, in each of which all the variables occur with different indices, and (2) to be divisible by the simplest of all alternating functions of the variables, viz., the difference-product. The set of equations

$$a_r x + b_r y + c_r z + \ldots + g_r u + h_r v = k_r \qquad (r = 0, 1, \ldots, n-1)$$

is then attacked, the method being—to take the difference-product of $a, b, \ldots h$,—denote by D what the expansion of this becomes when exponents are changed into suffixes,—denote by A_r the co-factor of a_r in D,—then obtain the equations

$$\begin{array}{l} A_0 a_0 + A_1 a_1 + A_2 a_2 + \ldots + A_{n-1} a_{n-1} = D\,, \\ A_0 b_0 + A_1 b_1 + A_2 b_2 + \ldots + A_{n-1} b_{n-1} = 0\,, \\ A_0 c_0 + A_1 c_1 + A_2 c_2 + \ldots + A_{n-1} c_{n-1} = 0\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ A_0 h_0 + A_1 h_1 + A_2 h_2 + \ldots + A_{n-1} h_{n-1} = 0\,, \end{array}$$

—and thereafter proceed as Laplace had taught. As in the memoir of 1812, the "symbolic" form of the values of $x, y, \ldots$ is unfailingly given.

A note is added (pp. 521–524) on the development of the difference-product, showing how all the terms may be got from one by interchanging one exponent with another, how the signs depend on the number of said interchanges, and how it may be ascertained whether any two given terms have like or unlike signs.

It will thus be seen that not only is the name "determinant" never mentioned in the chapter, and the notation $S \pm a_0 b_1 c_2 \ldots h_{n-1}$ never used, but that the subject is scarcely so much as touched upon. Although, therefore, Cauchy's text-book went through a considerable number of editions, and had a widespread influence, it gave no such impulse as it might have done to the study of the theory of determinants.

SCHERK (1825).

[Mathematische Abhandlungen. Von Dr. Heinrich Ferdinand Scherk, . . . iv. + 116 pp. Berlin. Pp. 31–66. Zweite Abhandlung: Allgemeine Auflösung der Gleichungen des ersten Grades mit jeder beliebigen Anzahl von unbekannten Grössen, und einige dahin gehörige analytische Untersuchungen.]

The only previous writings of importance known to Scherk were, according to his own statement, those of Cramer, Bézout (1764), Vandermonde, Bézout (1779), Hindenburg, and Rothe. His style bears most resemblance to Rothe's, whose paper, however, he does not speak of with unmixed eulogy, characterising it as containing "eine strenge aber ziemlich weitläuftige Auflösung der Aufgabe."

The main part of the memoir consists of a lengthy demonstration, extending, indeed, to 17 pages quarto, of Cramer's rule, or rather of Cramer's set of three rules (iv., v., iii. 2), by the method of so-called mathematical induction. The peculiarity of the demonstration is that it is entered upon without any previous examination of the properties of Cramer's functions (determinants); and it is noteworthy on two grounds—(1) as being new, and (2) because the properties, which it really if not explicitly employs, had also not been previously referred to.

The cases of one equation with one unknown, two equations with two unknowns, three equations with three unknowns, are dealt with in succession, the solution of one case being used in obtaining the solution of the next. All three solutions are noted as being in accordance with Cramer's rules, and the said rules being formulated, and supposed to hold,—for n equations with n unknowns, it is sought to establish their validity for $n+1$ equations with $n+1$ unknowns. In other words, the set of n equations being

$$\left.\begin{array}{l}
\overset{n}{\underset{1}{a}}\overset{n}{x} + \overset{n-1}{\underset{1}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{1}{a}}\overset{n-2}{x} + \ldots\ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} \\
\overset{n}{\underset{2}{a}}\overset{n}{x} + \overset{n-1}{\underset{2}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{2}{a}}\overset{n-2}{x} + \ldots\ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\overset{n}{\underset{n}{a}}\overset{n}{x} + \overset{n-1}{\underset{n}{a}}\overset{n-1}{x} + \overset{n-2}{\underset{n}{a}}\overset{n-2}{x} + \ldots\ldots + \overset{1}{\underset{n}{a}}\overset{1}{x} = \underset{n}{s}
\end{array}\right\}$$

and the corresponding values of

$$\overset{1}{x},\ \overset{2}{x},\ \ldots\ldots\ \overset{n}{x},$$

being

$$\frac{P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\ \overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\ \overset{1}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right)},\quad \frac{P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\ \overset{2}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\ \overset{2}{\underset{h}{a}},\ \overset{2}{\underset{h}{a}}\right)},\quad \ldots\ldots,\quad \frac{P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\ \overset{n}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}};\ \overset{n}{\underset{h}{a}},\ \overset{n}{\underset{h}{a}}\right)},$$

it is required to show that the solution of the set of $n+1$ equations

$$\left.\begin{array}{l}
\overset{n+1}{\underset{1}{a}}\overset{n+1}{x} + \overset{n}{\underset{1}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} \\
\overset{n+1}{\underset{2}{a}}\overset{n+1}{x} + \overset{n}{\underset{2}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{2}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\overset{n+1}{\underset{n+1}{a}}\overset{n+1}{x} + \overset{n}{\underset{n+1}{a}}\overset{n}{x} + \ldots + \overset{k}{\underset{n+1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{n+1}{a}}\overset{1}{x} = \underset{n+1}{s}
\end{array}\right\}$$

is

$$\overset{1}{x} = \frac{P\left(\overset{n+1}{\underset{n+1}{a}};\ \underset{h}{s},\ \overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n+1}{\underset{n+1}{a}};\ \overset{1}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right)},\ \ldots\ldots,\ \overset{n+1}{x} = \frac{\left(\overset{n+1}{\underset{n+1}{a}};\ \underset{h}{s},\ \overset{n+1}{\underset{h}{a}}\right)}{\left(\overset{n+1}{\underset{n+1}{a}};\ \overset{n+1}{\underset{h}{a}},\ \overset{n+1}{\underset{h}{a}}\right)}. \quad \text{(XIII. 4.)}$$

Before proceeding, the notation

$$P\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\ \overset{1}{\underset{h}{a}}\right)$$

requires attention. It is meant to be an epitome of Cramer's rules; the first half of the group of symbols, viz. $P\left(\overset{n}{\underset{n}{a}}\right.$ implying permutation of the under-indices of the product $\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}} \ldots \overset{n}{\underset{n}{a}}$ and aggregation of the different products thus obtained, each taken with its proper sign: and the second half implying that in every term of this aggregate $\underset{h}{s}$ is to be substituted for $\underset{h}{a}$. A modern writer would denote the same thing by

$$\left|\begin{array}{ccccc}
\underset{1}{s} & \overset{2}{\underset{1}{a}} & \overset{3}{\underset{1}{a}} & \ldots & \overset{n}{\underset{1}{a}} \\
\underset{2}{s} & \overset{2}{\underset{2}{a}} & \overset{3}{\underset{2}{a}} & \ldots & \overset{n}{\underset{2}{a}} \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\underset{n}{s} & \overset{2}{\underset{n}{a}} & \overset{3}{\underset{n}{a}} & \ldots & \overset{n}{\underset{n}{a}}
\end{array}\right|,$$

only it must be noted that in using $P\left(\overset{n}{\underset{n}{a}}\,;\ \underset{h}{s}\,,\ \overset{1}{\underset{h}{a}}\right)$ at this stage, we leave out of account the signs of the terms composing it, the rule of signs being the subject of a separate investigation. Any one of the forms

$$P\left(\overset{n}{\underset{n}{a}}\,;\ \overset{1}{\underset{h}{a}}\,,\ \overset{1}{\underset{h}{a}}\right),\quad P\left(\overset{n}{\underset{n}{a}}\,;\ \overset{2}{\underset{h}{a}}\,,\ \overset{2}{\underset{h}{a}}\right),\ \ldots\ldots$$

it need scarcely be added, will thus stand for the common denominator.

Of the $n+1$ equations the first n are taken, written in the form

$$\left.\begin{array}{l}
\overset{n}{\underset{1}{a}}\overset{n}{x} + \overset{n-1}{\underset{1}{a}}\overset{n-1}{x} + \ldots + \overset{k}{\underset{1}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{1}{a}}\overset{1}{x} = \underset{1}{s} - \overset{n+1}{\underset{1}{a}}\overset{n+1}{x} \\
\overset{n}{\underset{2}{a}}\overset{n}{x} + \overset{n-1}{\underset{2}{a}}\overset{n-1}{x} + \ldots + \overset{k}{\underset{2}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{2}{a}}\overset{1}{x} = \underset{2}{s} - \overset{n+1}{\underset{2}{a}}\overset{n+1}{x} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\overset{n}{\underset{n}{a}}\overset{n}{x} + \overset{n-1}{\underset{n}{a}}\overset{n-1}{x} + \ldots + \overset{k}{\underset{n}{a}}\overset{k}{x} + \ldots + \overset{1}{\underset{n}{a}}\overset{1}{x} = \underset{n}{s} - \overset{n+1}{\underset{n}{a}}\overset{n+1}{x}
\end{array}\right\}$$

and solved, the results being by hypothesis

$$\overset{1}{x} = \frac{P\left(\overset{n}{\underset{n}{a}}\,;\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x}\,,\ \overset{1}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}}\,;\ \overset{1}{\underset{h}{a}}\,,\ \overset{1}{\underset{h}{a}}\right)},$$

$$\cdots\cdots\cdots\cdots$$

$$\overset{k}{x} = \frac{P\left(\overset{n}{\underset{n}{a}}\,;\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x}\,,\ \overset{k}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}}\,;\ \overset{2}{\underset{h}{a}}\,,\ \overset{2}{\underset{h}{a}}\right)},$$

$$\cdots\cdots\cdots\cdots$$

$$\overset{n}{x} = \frac{P\left(\overset{n}{\underset{n}{a}}\,;\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x}\,,\ \overset{n}{\underset{h}{a}}\right)}{P\left(\overset{n}{\underset{n}{a}}\,;\ \overset{n}{\underset{h}{a}}\,,\ \overset{n}{\underset{h}{a}}\right)}.$$

These values are then of course substituted in the $(n+1)^{\text{th}}$ equation, which thus becomes

$$\overset{n+1}{\underset{n+1}{a}}\overset{n+1}{x} + \overset{n}{\underset{n+1}{a}}\,\frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\ \overset{1}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \overset{1}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right)} + \ldots + \overset{k}{\underset{n+1}{a}}\,\frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\ \overset{k}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right)}$$

$$+ \ldots + \overset{1}{\underset{n+1}{a}}\,\frac{\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\ \overset{1}{\underset{h}{a}}\right)}{\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \overset{n}{\underset{h}{a}},\ \overset{n}{\underset{h}{a}}\right)} = \underset{n+1}{s}\ ;$$

and as this manifestly involves none of the unknowns but $\overset{n+1}{x}$, the object must now be to solve for $\overset{n+1}{x}$, and then show what the value obtained is transformable into. The way in which this is effected is well worthy of attention. Scherk's own words in regard to the first steps are (p. 40)—

"Da aber $\underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x}$ in jeder einzelnen Permutationsform nur Einmal, nämlich in der ersten Potenz vorkommt, so bedeutet das Zeichen

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\ \overset{k}{\underset{h}{a}}\right)$$

dass in jede der in I. beschriebenen Permutationsformen für $\overset{k}{\underset{h}{a}}$ erst $\underset{h}{s}$, dann $\overset{n+1}{\underset{h}{a}}\overset{n+1}{x}$ gesetzt, und beide Resultate von einander abgezogen werden sollen: folglich ist

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\ \overset{k}{\underset{h}{a}}\right) = \mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s},\ \overset{k}{\underset{h}{a}}\right) - \mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\ \overset{k}{\underset{h}{a}}\right).$$

In dem letzten Gliede dieser Gleichung kömmt aber in jeder Form $\overset{n+1}{x}$, und zwar zur ersten Potenz, vor; $\overset{n+1}{x}$ ist also gemeinschaftlicher Factor aller Formen, und folglich ist

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \underset{h}{s} - \overset{n+1}{\underset{h}{a}}\overset{n+1}{x},\ \overset{k}{\underset{h}{a}}\right) = \mathrm{P}\left(\overset{n}{\underset{h}{a}};\ \underset{h}{s},\ \overset{k}{\underset{h}{a}}\right) - \overset{n+1}{x}\,\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \overset{n+1}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right).$$

Macht man diese Substitution für $k = 1, 2, \ldots, n$, in der letzten Gleichung, und bemerkt, dass

$$\mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \overset{1}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\right) = \mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \overset{2}{\underset{h}{a}},\ \overset{2}{\underset{h}{a}}\right) = \ldots\ldots = \mathrm{P}\left(\overset{n}{\underset{n}{a}};\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\right),$$

so geht diese in folgende Gleichung über

$$\underset{n+1}{\overset{n+1}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{k}{a}}\,,\,\underset{h}{\overset{k}{a}}\right)\overset{n+1}{x}$$

$$+\left\{\underset{n+1}{\overset{n}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{s}\,,\,\underset{h}{\overset{n}{a}}\right)+\ldots+\underset{n+1}{\overset{k}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{s}\,,\,\underset{h}{\overset{k}{a}}\right)+\ldots+\underset{n+1}{\overset{1}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{s}\,,\,\overset{1}{a}\right)\right\}$$

$$-\left\{\underset{n+1}{\overset{n}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{n+1}{a}}\,,\,\underset{h}{\overset{n}{a}}\right)+\ldots+\underset{n+1}{\overset{k}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{n+1}{a}}\,,\,\underset{h}{\overset{k}{a}}\right)+\ldots+\underset{n+1}{\overset{1}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{n+1}{a}}\,,\,\underset{h}{\overset{1}{a}}\right)\right\}\overset{n+1}{x}$$

$$=\underset{n+1}{s}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{k}{a}}\,,\,\underset{h}{\overset{k}{a}}\right);$$

folglich

$$\overset{n+1}{x}=\frac{-\underset{n+1}{\overset{1}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{s}\,,\,\underset{h}{\overset{1}{a}}\right)-\underset{n+1}{\overset{2}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{s}\,,\,\underset{h}{\overset{2}{a}}\right)-\ldots-\underset{n+1}{\overset{n}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{s}\,,\,\underset{h}{\overset{n}{a}}\right)+\underset{n+1}{s}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{k}{a}}\,,\,\underset{h}{\overset{k}{a}}\right)}{-\underset{n+1}{\overset{1}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{n+1}{a}}\,,\,\underset{h}{\overset{1}{a}}\right)-\underset{n+1}{\overset{2}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{n+1}{a}}\,,\,\underset{h}{\overset{2}{a}}\right)-\ldots-\underset{n+1}{\overset{n}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{n+1}{a}}\,,\,\underset{h}{\overset{n}{a}}\right)+\underset{n+1}{\overset{n+1}{a}}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{k}{a}}\,,\,\underset{h}{\overset{k}{a}}\right)}.$$

The first theorem here made use of and formulated, viz.,

$$\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{s}-\underset{h}{\overset{n+1}{a}}\,\overset{n+1}{x}\,,\,\underset{h}{\overset{k}{a}}\right)=\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{s}\,,\,\underset{h}{\overset{k}{a}}\right)-\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{n+1}{a}}\,\overset{n+1}{x}\,,\,\underset{h}{\overset{k}{a}}\right)\qquad(\text{XLVII.})$$

is the now familiar rule for the partition of a determinant with a row or column of binomial elements into two determinants, or for the addition of two determinants which are identical except in one row or one column. The second theorem, viz.,

$$\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{n+1}{a}}\,\overset{n+1}{x}\,,\,\underset{h}{\overset{k}{a}}\right)=\overset{n+1}{x}\,\mathrm{P}\left(\underset{n}{\overset{n}{a}}\,;\,\underset{h}{\overset{n+1}{a}}\,,\,\underset{h}{\overset{k}{a}}\right)\qquad(\text{XLVIII.})$$

is the now equally familiar theorem regarding the multiplication of a determinant by means of the multiplication of all the elements of a row or column. That these two very elementary theorems should not have been noted until the time of Scherk is rather remarkable.

The consideration of the constitution of

$$\mathrm{P}\left(\underset{n+1}{\overset{n+1}{a}}\,;\,\underset{h}{\overset{k}{a}}\,,\,\underset{h}{\overset{k}{a}}\right)$$

is next entered upon, with the object of showing that the terms are exactly the terms of the denominator

$$-\overset{1}{\underset{n+1}{a}}\,\mathrm{P}\begin{pmatrix}n & n+1 & 1\\ a\,; & a & , a\\ n & h & h\end{pmatrix}-\overset{2}{\underset{n+1}{a}}\,\mathrm{P}\begin{pmatrix}n & n+1 & 2\\ a\,; & a & , a\\ n & h & h\end{pmatrix}-\ldots\ldots+\overset{n+1}{\underset{n+1}{a}}\,\mathrm{P}\begin{pmatrix}n & k & k\\ a\,; & a & , a\\ n & h & h\end{pmatrix}.$$

More than two pages are occupied with this part proof of Bézout's recurrent law of formation. The identity of the terms of

$$\mathrm{P}\begin{pmatrix}n+1 & & n+1\\ a & ;\ s\,, & a\\ n+1 & h & h\end{pmatrix}$$

with the terms of the numerator then follows at once; and the desired form for the value of $\overset{n+1}{x}$, so far as the *magnitude* of the terms is concerned, is thus obtained. The corresponding forms for $x_1, x_2, \ldots$ are of course immediately deducible.

The rules for obtaining the terms of the numerator and denominator having been thus established in all their generality, the rule of signs is next dealt with. The treatment is cumbersome, but fresh and interesting. It is pointed out, to start with, that the counting of the inversions of order of a permutation, is equivalent to subtracting separately from each element all the elements which follow it, reckoning $+1$ as a sign-factor when the difference is positive, and -1 when the difference is negative, and then taking the product of all the said factors. This, it will be recalled, is essentially identical with an observation of Cauchy's. Scherk, however, goes on to remark that these sign-factors may be viewed as functions of the differences which give rise to them, and may be so represented. Whether there actually be a function which equals $+1$ for all positive values of the argument and equals -1 for all negative values is left for future consideration. Cramer's rule of signs is thus made to take the following form (p. 45):—

> "Wenn $\phi(\beta)$ eine solche Function der ganzen Zahl β ist, welche $= +1$ ist, wenn β positiv, und -1, wenn β negativ ist, so ist das Vorzeichen Z irgend eines in dem Aggregate
>
> $\mathrm{P}\begin{pmatrix}n & h & h\\ a\,; & a & , a\\ n & k & k\end{pmatrix}$ enthaltenen Gliedes
>
> $$\underset{\alpha}{\overset{1}{a}}\,\underset{\alpha}{\overset{2}{\dot a}}\,\underset{\alpha'''}{\overset{3}{a}}\ldots\underset{\alpha^{(k-1)}}{\overset{k-1}{a}}\quad\underset{\alpha^{(k)}}{\overset{k}{a}}\quad\underset{\alpha^{(k+1)}}{\overset{k-1}{a}}\quad\ldots\underset{\alpha^{(n)}}{\overset{n}{a}}$$
>
> folgendes:

$$
\begin{aligned}
Z = {} & \phi(a'' - a').\phi(a''' - a')\ldots\phi(a^{(k-1)} - a').\phi(a^{(k)} - a').\phi(a^{(k+1)} - a')\ldots\phi(a^{(n)} - a') \\
& \phi(a''' - a'').\phi(a'''' - a'')\ldots\phi(a^{(k)} - a'').\phi(a^{(k+1)} - a'').\phi(a^{(k+2)} - a'')\ldots\phi(a^{(n)} - a'') \\
& \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
& \phi(a^{(k-1)} - a^{(k-2)}).\ \phi(a^{(k)} - a^{(k-2)})\ldots\ \phi(a^{(n)} - a^{(k-2)}) \\
& \phi(a^{(k)} - a^{(k-1)}).\ \phi(a^{(k+1)} - a^{(k-1)})\ldots\ \phi(a^{(n)} - a^{(k-1)}) \\
& \phi(a^{(k+1)} - a^{(k)}).\ \phi(a^{(k+2)} - a^{(k)})\ \ldots\ \phi(a^{(n)} - a^{(k)}) \\
& \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
& \phi(a^{(n)} - a^{(n-1)}).
\end{aligned}
\qquad \text{(III. 22.)}
$$

And it is this form which Scherk seeks to establish. The mode of proof is again the so-called inductive mode. In the case of *two* permutable indices the law is readily seen to hold. We thus have, preparatory for the next case,

$$
\mathrm{P}\Big(\overset{2}{\underset{2}{a}}\,;\ \overset{k}{\underset{h}{a}}\,,\ \overset{k}{\underset{h}{a}}\Big) = \phi(2-1)\,\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\,\overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}},
$$

$$
\mathrm{P}\Big(\overset{2}{\underset{2}{a}}\,;\ \overset{3}{\underset{h}{a}}\,,\ \overset{1}{\underset{h}{a}}\Big) = \phi(2-1)\,\overset{3}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\,\overset{3}{\underset{2}{a}}\overset{2}{\underset{1}{a}},
$$

$$
\mathrm{P}\Big(\overset{2}{\underset{2}{a}}\,;\ \overset{3}{\underset{h}{a}}\,,\ \overset{2}{\underset{h}{a}}\Big) = \phi(2-1)\,\overset{1}{\underset{1}{a}}\overset{3}{\underset{2}{a}} + \phi(1-2)\,\overset{1}{\underset{2}{a}}\overset{3}{\underset{1}{a}}.
$$

But "nach dem Obigem"

$$
\mathrm{P}\Big(\overset{3}{\underset{3}{a}}\,;\ \overset{k}{\underset{h}{a}}\,,\ \overset{k}{\underset{h}{a}}\Big) = -\overset{1}{\underset{3}{a}}\mathrm{P}\Big(\overset{2}{\underset{2}{a}}\,;\ \overset{3}{\underset{h}{a}},\ \overset{1}{\underset{h}{a}}\Big) - \overset{2}{\underset{3}{a}}\mathrm{P}\Big(\overset{2}{\underset{2}{a}}\,;\ \overset{3}{\underset{h}{a}},\ \overset{2}{\underset{h}{a}}\Big) + \overset{3}{\underset{3}{a}}\mathrm{P}\Big(\overset{2}{\underset{2}{a}}\,;\ \overset{k}{\underset{h}{a}},\ \overset{k}{\underset{h}{a}}\Big).
$$

Consequently

$$
\begin{aligned}
\mathrm{P}\Big(\overset{3}{\underset{3}{a}}\,;\ \overset{k}{\underset{h}{a}}\,,\ \overset{k}{\underset{h}{a}}\Big) = {} & -\overset{1}{\underset{3}{a}}\Big\{\phi(2-1)\overset{3}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{3}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\Big\} \\
& -\overset{2}{\underset{3}{a}}\Big\{\phi(2-1)\overset{1}{\underset{1}{a}}\overset{3}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{3}{\underset{1}{a}}\Big\} \\
& -\overset{3}{\underset{3}{a}}\Big\{\phi(2-1)\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}} + \phi(1-2)\overset{1}{\underset{2}{a}}\overset{2}{\underset{1}{a}}\Big\}.
\end{aligned}
$$

But as
$$
\begin{aligned}
-\phi(2-1) &= \phi(1-2) \\
\text{and}\qquad -1 &= \phi(2-3) \\
\text{and}\qquad -1 &= \phi(1-3),
\end{aligned}
$$

we may substitute

$$
\phi(1-2)\phi(2-3)\phi(1-3) \quad \text{for} \quad -\phi(2-1).
$$

This and five other similar substitutions give us

$$\begin{aligned}
\mathrm{P}\left(\underset{3}{\overset{3}{a}};\ \underset{h}{\overset{k}{a}},\ \underset{h}{\overset{k}{a}}\right) &= \phi(2-1)\phi(3-1)\phi(3-2)\ \underset{1}{\overset{1}{a}}\underset{2}{\overset{2}{a}}\underset{3}{\overset{3}{a}} + \phi(3-1)\phi(2-1)\phi(2-3)\ \underset{1}{\overset{1}{a}}\underset{3}{\overset{2}{a}}\underset{2}{\overset{3}{a}} \\
&+ \phi(1-2)\phi(3-2)\phi(3-1)\ \underset{2}{\overset{1}{a}}\underset{1}{\overset{2}{a}}\underset{3}{\overset{3}{a}} + \phi(3-2)\phi(1-2)\phi(1-3)\ \underset{2}{\overset{1}{a}}\underset{3}{\overset{2}{a}}\underset{1}{\overset{3}{a}} \\
&+ \phi(1-3)\phi(2-3)\phi(2-1)\ \underset{3}{\overset{1}{a}}\underset{1}{\overset{2}{a}}\underset{2}{\overset{3}{a}} + \phi(2-3)\phi(1-3)\phi(1-2)\ \underset{3}{\overset{1}{a}}\underset{2}{\overset{2}{a}}\underset{1}{\overset{3}{a}}\ ;
\end{aligned}$$

so that the law is seen to hold also for the case of *three* permutable indices. The completion of the proof, giving the transition from n to $n+1$ permutable indices, occupies three pages.

This is followed by two pages devoted to the subjects temporarily set aside at the outset, viz., the possible existence of functions having the peculiar properties of ϕ. Two amusing instances of such functions are given,—

$$(1)\qquad \phi(\beta) = \left.\begin{aligned} &\mathrm{P}_0^{\beta-1} + \mathrm{P}_0^{\beta-2} + \mathrm{P}_0^{\beta-3} + \ \ldots \\ &- \mathrm{P}_0^{\beta-1} - \mathrm{P}_0^{\beta-2} - \mathrm{P}_0^{\beta-3} - \ \ldots\ldots \end{aligned}\right\}$$

$$(2)\qquad \begin{aligned} \phi(\beta) &= \left.\begin{aligned} &\frac{\sin 2\beta\pi}{(\beta-1)2\pi} + \frac{\sin 2\beta\pi}{(\beta-2)2\pi} + \frac{\sin 2\beta\pi}{(\beta-3)2\pi} + \ldots\ldots \\ &- \frac{\sin 2\beta\pi}{(\beta+1)2\pi} - \frac{\sin 2\beta\pi}{(\beta+2)2\pi} - \frac{\sin 2\beta\pi}{(\beta+3)2\pi} - \ldots\ldots \end{aligned}\right\} \\ &= \left(\frac{1}{\beta^2-1} + \frac{2}{\beta^2-4} + \frac{3}{\beta^2-9} + \ldots\right)\frac{\sin 2\beta\pi}{\pi}. \end{aligned}$$

where P_0^k stands for the k^{th} coefficient in the expansion of $(a+b)^0$. Success, however far from brilliant it may be, in thus expressing the rule of signs by means of the symbols of analyses, led Scherk to try to do the same for the rule of formation of the terms. Nothing came of the attempt, however. "Bald aber," he says, "zeigte es sich dass Permutationen niemals durch andere analytische Zeichen ersetzt werden könnten."

Such speculations are not altogether uninteresting when later work like Hankel's comes to be considered.

In an Appendix dealing (1) with the case of a set of linear equations which are not all independent, (2) with the solution of particular sets of equations, there is given at the outset a proof of

the theorem regarding the sign of a permutation which is got from another permutation by the interchange of two elements. If the under-indices of the one term whose sign is z be

$$a'a''a''' \ldots . \, a^{(i-1)}a^{(i)}a^{(i+1)} \ldots .. \, a^{(k-1)}a^{(k)}a^{(k+1)} \ldots . \, a^{(n)},$$

and of the other whose sign is Z * be

$$a'a''a''' \ldots . \, a^{(i-1)}a^{(k)}a^{(i+1)} \ldots .. \, a^{(k-1)}a^{i}a^{(k+1)} \ldots . \, a^{(n)}$$

it is shown that

$$\frac{z}{\mathrm{Z}} = \frac{\phi(a^{i+1} - a^{i})}{\phi(a^{i} - a^{i+1})} \cdot \frac{\phi(a^{i+2} - a^{i})}{\phi(a^{i} - a^{i+2})} \cdots \frac{\phi(a^{k} - a^{i})}{\phi(a^{i} - a^{k})}$$

$$\times \frac{\phi(a^{k} - a^{i})}{\phi(a^{i} - a^{k})} \cdot \frac{\phi(a^{k} - a^{i+1})}{\phi(a^{i+1} - a^{k})} \cdots \frac{\phi(a^{k} - a^{k-1})}{\phi(a^{k-1} - a^{k})};$$

and there being here $2k - 2i - 1$ quotients each $= -1$, the result arrived at is

$$\frac{z}{\mathrm{Z}} = -1 \qquad \text{or} \qquad z = -\mathrm{Z},$$

as was to be proved. (III. 23.)

The body of the Appendix contains, along with other matter which falls to be considered later, the statement and proof of propositions identical in essence but not in form with the following:—

$$(1) \qquad \begin{vmatrix} \overset{1}{\underset{1}{a}} & \overset{2}{\underset{1}{a}} & \ldots & \overset{n-1}{\underset{1}{a}} & \overset{n}{\underset{1}{a}} \\ \overset{1}{\underset{2}{a}} & \overset{2}{\underset{2}{a}} & \ldots & \overset{n-1}{\underset{2}{a}} & \overset{n}{\underset{2}{a}} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \overset{1}{\underset{n-1}{a}} & \overset{2}{\underset{n-1}{a}} & \ldots & \overset{n-1}{\underset{n-1}{a}} & \overset{n}{\underset{n-1}{a}} \\ \overset{1}{\mathrm{T}} & \overset{2}{\mathrm{T}} & \ldots & \overset{n-1}{\mathrm{T}} & \overset{n}{\mathrm{T}} \end{vmatrix} \qquad \text{(XLIX.)}$$

where

$$\overset{1}{\mathrm{T}} = \underset{1}{m}\overset{1}{\underset{1}{a}} + \underset{2}{m}\overset{1}{\underset{2}{a}} + \ldots . + \underset{n-1}{m}\overset{1}{\underset{n-1}{a}}$$

$$\overset{2}{\mathrm{T}} = \underset{1}{m}\overset{2}{\underset{1}{a}} + \underset{2}{m}\overset{2}{\underset{2}{a}} + \ldots . + \underset{n-1}{m}\overset{2}{\underset{n-1}{a}}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

* More than a page is occupied in writing the expressions for z and Z.

$$(2)\qquad \begin{vmatrix} \overset{1}{\underset{1}{a}} & & & & & \\ \overset{1}{\underset{2}{a}} & \overset{2}{\underset{2}{a}} & & & \vdots & \\ \overset{1}{\underset{3}{a}} & \overset{2}{\underset{3}{a}} & \overset{3}{\underset{3}{a}} & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \overset{1}{\underset{n}{a}} & \overset{2}{\underset{n}{a}} & \overset{3}{\underset{n}{a}} & \cdots & \cdots & \overset{n}{\underset{n}{a}} \end{vmatrix} = \overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}}\cdots\cdots\overset{n}{\underset{n}{a}}$$

The first of these is proved from first principles, and not by the immediate use of theorems XLVII., XLVIII. above. The second is proved by noting that any other term is got from the first

$$\overset{1}{\underset{1}{a}}\overset{2}{\underset{2}{a}}\overset{3}{\underset{3}{a}}\cdots\cdots\overset{n}{\underset{n}{a}}$$

by permutation of the under-indices, that any such permutation will introduce one or more elements whose upper-index exceeds the lower, and that such are all zero. (VI. 5.)

SCHWEINS (1825).

[Theorie der Differenzen und Differentiale, u.s.w. Von Ferd. Schweins. vi. + 666 pp. Heidelberg, 1825. Pp. 317–431; Theorie der Producte mit Versetzungen.]

With much of the preceding literature, Schweins, our next author, was thoroughly familiar. Cramer, Bézout, Hindenburg, Rothe, Laplace, Desnanot, and Wronski he refers to by name. The one notable investigator left out of his list is Cauchy, whose important memoir bearing date 1812 might have been known, one would think, to a writer who knew Desnanot's book of 1819 and Wronski's memoirs of 1810, 1811, &c. Still more curious is the omission of Vandermonde's name, whose memoir, as we have seen, is to be found in the very same volume as that of Laplace.

Schweins' portly volume consists of seven separate treatises. It is the third, headed *Theorie der Producte mit Versetzungen*, which deals expressly and exclusively with the subject of deter-

minants. The treatise is logically arranged and carefully written. It opens with an introduction of 4 pp., the main part of which serves as a table of contents and as a guide to the theorems which the author considered his own. It consists of four Sections (Abtheilungen), subdivided into portions which we may call chapters, the first section containing five chapters, the second also five, the third one, and the fourth four.

Schweins' name for the functions is

Producte mit Versetzungen; (xv. 6.)

his notation is a modification of Laplace's, viz., he uses

$$\Big\| \qquad \Big) \qquad \text{(vii. 6)}$$

where Laplace used simply

$$\Big(\qquad \Big);$$

and his definition is the same as Vandermonde's; that is to say, he employs Bézout's law of recurring formation. His words at the outset are—

"Die Bildungsweise der Producte, welche hier untersucht werden sollen, geben folgende Zeichen an :—

$$\left\|\begin{matrix}a_1\\A_1\end{matrix}\right) = \overset{a_1}{A_1},$$

$$\left\|\begin{matrix}a_1&a_2\\A_1&A_2\end{matrix}\right) = \left\|\begin{matrix}a_1\\A_1\end{matrix}\right).\overset{a_2}{A_2} - \left\|\begin{matrix}a_2\\A_1\end{matrix}\right).\overset{a_1}{A_2},$$

$$\left\|\begin{matrix}a_1&a_2&a_3\\A_1&A_2&A_3\end{matrix}\right) = \left\|\begin{matrix}a_1&a_2\\A_1&A_2\end{matrix}\right).\overset{a_3}{A_3} - \left\|\begin{matrix}a_1&a_3\\A_1&A_2\end{matrix}\right).\overset{a_2}{A_3} + \left\|\begin{matrix}a_2&a_3\\A_1&A_2\end{matrix}\right).\overset{a_1}{A_3},$$

$$\left\|\begin{matrix}a_1&a_2&a_3&a_4\\A_1&A_2&A_3&A_4\end{matrix}\right) = \left\|\begin{matrix}a_1&a_2&a_3\\A_1&A_2&A_3\end{matrix}\right).\overset{a_4}{A_4} - \left\|\begin{matrix}a_1&a_2&a_4\\A_1&A_2&A_3\end{matrix}\right).\overset{a_3}{A_4}$$
$$+ \left\|\begin{matrix}a_1&a_3&a_4\\A_1&A_2&A_3\end{matrix}\right).\overset{a_2}{A_4} - \left\|\begin{matrix}a_2&a_3&a_4\\A_1&A_2&A_3\end{matrix}\right).\overset{a_1}{A_4},$$

und allgemein

$$\left\|\begin{matrix}a_1\cdots\cdots a_n\\A_1\ldots\ldots A_n\end{matrix}\right) = (-)\left\|\begin{matrix}a_1\cdots\cdots a_{n-1}\\A_1\ldots\ldots A_{n-1}\end{matrix}\right).\overset{a_n}{A_n} + (-)^1\left\|\begin{matrix}a_1\cdots a_{n-2}\,a_n\\A_1\ldots\ldots A_{n-1}\end{matrix}\right)\overset{a_{n-1}}{A_n} + \ldots$$
$$+ (-)^x\left\|\begin{matrix}a_1\cdots a_{n-x-1}\,a_{n-x+1}\cdots a_n\\A_1\ldots\ldots\ldots\ldots A_{n-1}\end{matrix}\right)\overset{a_{n-x}}{A_n} + \ldots + (-)^{n-1}\left\|\begin{matrix}a_2\cdots\cdots a_n\\A_1\ldots A_{n-1}\end{matrix}\right)\overset{a_1}{A_n},$$

oder

$$\left\|\begin{matrix}a_1\cdots\cdots a_n\\A_1\ldots\ldots A_n\end{matrix}\right) = \Sigma(-)^x\left\|\begin{matrix}a_1\cdots a_{n-x-1}\,a_{n-x+1}\cdots a_n\\A_1\ldots\ldots\ldots\ldots A_{n-1}\end{matrix}\right).\overset{a_{n-x}}{A_n} \qquad x=0,1,\ldots,n-1.$$

The sequence of propositions as might be expected is not unlike that found in Vandermonde. The first six propositions are—

1. The under elements (A_1, A_2, &c.) being allowed to remain unchanged, the upper elements (a_1, a_2, . . .) are interchanged in every possible way to obtain the full development.

2. The sign preceding each term is dependent upon the number of interchanges of elements necessary to arrive at the term.

3. If two adjacent upper elements be interchanged, the sign of the *determinant* is altered.

4. If an upper element be moved a number of places to the right or left, the sign of the determinant is changed or not according as the number of places is odd or even.

5. If several upper elements change places, the sign of the determinant is altered or not according as the number is odd or even, which indicates how many cases there are of an element following one which in the original order it preceded.

6. If in any *term* the said number of pairs of elements in reversed order be even, the sign preceding the term must be positive; and if the number be odd, the sign must be negative.

The proof of the 3rd of these, which gave trouble to Vandermonde, is easily effected in what after all is Vandermonde's way, viz., by showing that the case for n elements follows with the help of the definition from the case for $n-1$ elements. (XI. 4.)

Schweins' 7th proposition is that there is an alternative recurring law of formation in which the under elements play the part of the upper elements in the original law, and *vice versa*. This amounts to saying in modern phraseology, that if a determinant has been shown to be developable in terms of the elements of a row and their complementary minors, it is also developable in terms of the elements of a column and their complementary minors. The proof is affected by the so-called method of induction, and is interesting both on its own account and from the fact that Cauchy's development in terms of binary products of a row and column turns up in the course of it. The character of the proof will be understood by the following illustrative example in the modern notation:—

By the original law of formation we have

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - a_2|b_1c_3d_4| + a_3|b_1c_2d_4| - a_4|b_1c_2d_3|;$$

and, as the new law is supposed to have been proved for determinants of the 3rd order, it follows that

$$\begin{aligned}|a_1b_2c_3d_4| = a_1|b_2c_3d_4| &- a_2\{b_1|c_3d_4| - c_1|b_3d_4| + d_1|b_3c_4|\} \\ &+ a_3\{b_1|c_2d_4| - c_1\,b_2d_4| + d_1|b_2c_4|\} \\ &- a_4\{b_1|c_2d_3| - c_1|b_2d_3| + d_1|b_2c_3|\}\,.\end{aligned}$$

Combining now by the original law the terms involving b_1 as a factor, the terms involving c_1, and those involving d_1, we obtain

$$|a_1b_2c_3d_4| = a_1|b_2c_3d_4| - b_1|a_2c_3d_4| + c_1|a_2b_3d_4| - d_1|a_2b_3c_4|\,;$$

and thus prove that the new law holds for determinants of the 4th order. (VI. 6.)

Cauchy's development above referred to appears in the penultimate identity in the convenient form of one term $a_1|b_2c_3d_4|$ followed by a square array of 9 terms. The form in Schweins' is—

$$\left|\begin{matrix}a_1 \cdots\cdots a_n \\ A_1 \ldots A_n\end{matrix}\right) = \left\|\begin{matrix}a_1 \cdots\cdots a_{n-1} \\ A_1 \ldots A_{n-1}\end{matrix}\right) \cdot \begin{matrix}a_n \\ A_n\end{matrix} + \Sigma_x\Sigma_y(-)^{x+y-1}\left\|\begin{matrix}a_1 \cdots\cdots a_{n-x-1} & a_{n-x+1} \cdots\cdots a_{n-1} \\ A_1 \ldots A_{n-y-1} & A_{n-y+1} \ldots A_{n-1}\end{matrix}\right) \cdot \begin{matrix}a_{n-x} \\ A_n\end{matrix} \cdot \begin{matrix}a_n \\ A_{n-y}\end{matrix}$$

Laplace's expansion-theorem is next taken up. To prepare the way a theorem in permutations is first given, the enunciation being as follows:—*If from* n *different elements every permutation of* q *elements be formed, and every permutation of* n − q *elements; and if each of the latter be appended to all such of the former as have no elements in common with it, all the permutations of the whole* n *elements will be obtained.* Thus, if the permutations of 1 2 3 4 5, or say P (1 2 3 4 5), be wanted, we first take the permutations three at a time, viz.,

P(1 2 3), P(1 2 4), P(1 2 5), ... , P(3 4 5)

where 1 2 3, 1 2 4, 1 2 5, . . ., 3 4 5 are the orderly arranged combinations of three elements; secondly, we take the permutations two at a time, viz.,

P(1 2), P(1 3), P(1 4), ... , P(4 5);

and, thirdly, we append each of the two permutations included in P(4 5) to each of the six included in P(1 2 3), each of the two in

P(3 5) to each of the six in P(1 2 4), and so on. The identity here involved Schweins writes as follows, the only difference being that P is put instead of V (*Versetzungen*) :—

$$\begin{aligned} P(1\,2\,3\,4\,5) = {} & P(1\,2\,3) \times P(4\,5) \\ & + P(1\,2\,4) \times P(3\,5) \\ & + P(1\,2\,5) \times P(3\,4) \\ & + P(1\,3\,4) \times P(2\,5) \\ & + P(1\,3\,5) \times P(2\,4) \\ & + P(1\,4\,5) \times P(2\,3) \\ & + P(2\,3\,4) \times P(1\,5) \\ & + P(2\,3\,5) \times P(1\,4) \\ & + P(2\,4\,5) \times P(1\,3) \\ & + P(3\,4\,5) \times P(1\,2) . \end{aligned}$$

Another example is—

$$\begin{aligned} P(1\,2\,3\,4\,5\,6) = {} & P(1\,2\,3) \,.\, P(4\,5\,6) \\ & + P(1\,2\,4) \,.\, P(3\,5\,6) \\ & \;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ & + P(3\,5\,6) \,.\, P(1\,2\,4) \\ & + P(4\,5\,6) \,.\, P(1\,2\,3) . \end{aligned}$$

The proof consists in the assertion that no permutation can occur twice on the right-hand side, and in showing that the number of permutations which occur is the full number.

From this lemma Laplace's expansion-theorem is given as an immediate deduction. The passage (p. 335) is interesting, as the mode of enunciating the theorem approximates closely to that of modern writers, and has a certain advantage over Cauchy's, perfectly accurate, more general and more compact though the latter be.

> "Nach dieser Weise, alle Versetzungen zu bilden, welche wir hier zuerst bekannt machen, können auch die Summen der Producte mit Versetzungen und mit veränderlichen Zeichen in niedrigere Summen zerlegt werden, wenn bei jeder Versetzung nach der oben gefundenen Vorschrift das zugehörige Zeichen bestimmt wird ; z. B.

$$
\begin{aligned}
\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & A_3 & A_4 & A_5 \end{matrix}\right)
&= \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_4 & A_5 \end{matrix}\right)
&&= \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_4 & A_5 \end{matrix}\right) \\
&- \left\|\begin{matrix} a_1 & a_2 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_3 & a_5 \\ A_4 & A_5 \end{matrix}\right)
&&- \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_4 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_3 & A_5 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_1 & a_2 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_3 & a_4 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_2 & A_5 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_3 & A_4 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_1 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_2 & a_5 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_3 & A_4 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_2 & A_5 \end{matrix}\right) \\
&- \left\|\begin{matrix} a_1 & a_3 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_2 & a_4 \\ A_4 & A_5 \end{matrix}\right)
&&- \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_3 & A_5 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_2 & A_4 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_1 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_2 & a_3 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_1 & A_4 & A_5 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_2 & A_3 \end{matrix}\right) \\
&- \left\|\begin{matrix} a_2 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_1 & a_5 \\ A_4 & A_5 \end{matrix}\right)
&&- \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_3 & A_4 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_5 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_2 & a_3 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_1 & a_4 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_3 & A_5 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_4 \end{matrix}\right) \\
&- \left\|\begin{matrix} a_2 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_1 & a_3 \\ A_4 & A_5 \end{matrix}\right)
&&- \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_2 & A_4 & A_5 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_3 \end{matrix}\right) \\
&+ \left\|\begin{matrix} a_3 & a_4 & a_5 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left\|\begin{matrix} a_1 & a_2 \\ A_4 & A_5 \end{matrix}\right)
&&+ \left\|\begin{matrix} a_1 & a_2 & a_3 \\ A_3 & A_4 & A_5 \end{matrix}\right) \cdot \left\|\begin{matrix} a_4 & a_5 \\ A_1 & A_2 \end{matrix}\right).
\end{aligned}
$$

In der ersten Scheitelreihe sind die oberen und in der zweiten die unteren Elemente veränderlich; die Zeichen + und − befolgen das Gesetz in § 140. Eben so ist

(Another example is given.)

Wir wollen für diese Bildungsweise folgende allgemeine Zeichen wählen:

$$
\left\|\begin{matrix} a_1 \cdots\cdots a_n \\ A_1 \cdots\cdots A_n \end{matrix}\right) = \Sigma(-)^* \left\|\begin{matrix} (a_1\, a_2 \cdots\cdots a_n)^{(q)} \\ A_1 \cdots\cdots A_q \end{matrix}\right) \cdot \left\|\begin{matrix} (a_1 \quad a_2 \quad \cdots\cdots a_n)^{(n-q)} \\ A_{q+1}\, A_{q+2} \cdots A_n \end{matrix}\right)
$$

und

$$
= \Sigma(-)^* \left\|\begin{matrix} a_1 \quad a_2 \cdots\cdots a_q \\ (A_1, A_2, \ldots, A_n)^{(q)} \end{matrix}\right) \cdot \left\|\begin{matrix} a_{q+1}\, a_{p+2} \cdots\cdots a_n \\ (A_1, A_2, \ldots, A_n)^{(n-q)} \end{matrix}\right)
$$

wo * nach dem Gesetze bestimmt werden muss, welches in § 140 gefunden ist." (XIV. 5.)

The one imperfection in this is in regard to the question of sign. It is implied that the sign to precede any product, say the product

$$\left|\begin{matrix} a_2 & a_3 & a_4 \\ A_1 & A_2 & A_3 \end{matrix}\right) \cdot \left|\begin{matrix} a_1 & a_5 \\ A_4 & A_5 \end{matrix}\right)$$

is fixed by making it the same as the sign of the *term*

$$\overset{a_2}{A_1}\,\overset{a_3}{A_2}\,\overset{a_4}{A_3}\,\overset{a_1}{A_4}\,\overset{a_5}{A_5}\,;$$

but nothing is said as to how this ensures that the 11 other terms of the product shall have their proper sign.

Considerably less interest attaches to the next theorem dealt with,—Vandermonde's theorem regarding the effect of the equality of two upper or two lower elements. All that is fresh is the lengthy demonstration by the method of so-called induction. The identities immediately following from it by expansion Schweins expresses as follows:—

$$\Sigma(-)^x \left\| \begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots a_{q-1}\, a_q\, a_{q+1} \ldots a_{n-1} \\ A_1 \ldots A_{x-1}\, A_{x+1} \ldots\ldots\ldots A_n \end{matrix} \right) \cdot \overset{a_q}{A_x} = 0$$

$$\Sigma(-)^x \left\| \begin{matrix} a_1 \ldots a_{x-1}\, a_{x+1} \ldots\ldots\ldots\ldots\ldots a_n \\ A_1 \ldots\ldots\ldots A_{q-1}\, A_q\, A_{q+1} \ldots A_{n-1} \end{matrix} \right) \cdot \overset{a_x}{A_q} = 0$$

where $x = 1, 2, \ldots\ldots, n$. (XII. 10.)

This concludes the first chapter of the first section.

The second chapter deals with a most notable generalisation, and is worthy of being reproduced with little or no abridgment. The subject may be described as the transformation of an aggregate of products of pairs of determinants into another aggregate of similar kind. A special example of the transformation is taken to open the chapter with, the initial aggregate of products being in this case

$$|a_1b_2c_3d_4|\cdot|e_5f_6g_7| - |a_1b_2c_3e_4|\cdot|d_5f_6g_7|$$
$$+ |a_1b_2c_3f_4|\cdot|d_5e_6g_7| - |a_1b_2c_3g_4|\cdot|d_5e_6f_7|\,.$$

Expanding the first factor of each product Schweins obtains

$$\left\{ d_4|a_1b_2c_3| - d_3|a_1b_2c_4| + d_2|a_1b_3c_4| - d_1|a_2b_3c_4| \right\}\cdot|e_5f_6g_7|$$
$$- \left\{ e_4|a_1b_2c_3| - e_3|a_1b_2c_4| + e_2|a_1b_3c_4| - e_1|a_2b_3c_4| \right\}\cdot|d_5f_6g_7|$$
$$+ \left\{ f_4|a_1b_2c_3| - f_3|a_1b_2c_4| + f_2|a_1b_3c_4| - f_1|a_2b_3c_4| \right\}\cdot|d_5e_6g_7|$$
$$- \left\{ g_4|a_1b_2c_3| - g_3|a_1b_2c_4| + g_2|a_1b_3c_4| - g_1|a_2b_3c_4| \right\}\cdot|d_5e_6f_7|\,.$$

He then combines the terms which contain $|a_1b_2c_3|$ as a factor, the terms which contain $|a_1b_2c_4|$ as a factor, and so forth, the result being by the law of formation,

$$|a_1b_2c_3|.|d_4e_5f_6g_7| - |a_1b_2c_4|.|d_3e_5f_6g_7| \\ + |a_1b_3c_4|.|d_2e_5f_6g_7| - |a_2b_3c_4|.|d_1e_5f_6g_7|.$$

The identity of this aggregate with the similar original aggregate constitutes the theorem.

The only point left in want of explanation in connection with it is the construction of the aggregate of products presented at the outset, it being, of course, impossible that any aggregate taken at will can be so transformable. A moment's examination suffices to show that when once the first product of all

$$|a_1b_2c_3d_4|.|e_5f_6g_7|$$

is chosen, the others are derivable from it in accordance with a simple law,—the requirements being (1) no change of suffixes, (2) the last letter of the first factor to be replaced by the letters of the second factor in succession, (3) the signs of the products to be + and − alternately. As for the first product of all, it is not difficult to see that the orders of the determinants composing it are quite immaterial. Instead of taking determinants of the 4th and 3rd orders, and producing by transformation an aggregate of products of determinants of the 3rd and 4th orders, we might have taken determinants of the $(n+1)^{\text{th}}$ and m^{th} orders, applied the transformation, and obtained an aggregate of products of determinants of the n^{th} and $(m+1)^{\text{th}}$ orders. This is the essence of Schweins' first generalisation. His own statement and proof of it leave little to be desired, and are worthy of examination in order that his firm grasp of the subject and his command of the notation may be known. He says (p. 345)—

"Die Reihe, welche in eine andere übertragen werden soll, sei

$$Q = \Sigma_x(-)^{x-1} \left\| \begin{matrix} a_1 \ldots\ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n\, B_x \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix} \right)$$

$$\text{wo} \quad x = 1, 2, \ldots\ldots, m+1\,.$$

Der erste Factor wird nach 515 in niedere Summen aufgelöst

$$\left\| \begin{matrix} a_1 \ldots\ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n\, B_x \end{matrix} \right) = \Sigma_y(-)^{n-y+1} \left\| \begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix} \right) \cdot \overset{a_y}{B_x}$$

$$\text{wo} \quad y = 1, 2, \ldots, n+1$$

wodurch die vorgegebene Reihe in folgende übergeht:

$$Q = \Sigma_x \Sigma_y (-)^{n+x-y} \left\| \begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix} \right) \cdot \begin{matrix} a_y \\ B_x \end{matrix} .$$

Es ist aber nach 522

$$\Sigma_x (-)^{m+1-x} \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix} \right) \cdot \begin{matrix} a_y \\ B_x \end{matrix} = \left\| \begin{matrix} b_1 \ldots\ldots b_m\, a_y \\ B_1 \ldots\ldots\ldots B_{m+1} \end{matrix} \right)$$

$$= (-)^m \left\| \begin{matrix} a_y\, b_1\, b_2 \ldots\ldots b_m \\ B_1 \ldots\ldots\ldots B_{m+1} \end{matrix} \right) ;$$

folglich

$$Q = \Sigma_y (-)^{n-y+1} \left\| \begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix} \right) . \left\| \begin{matrix} a_y\, b_1\, b_2 \ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right) ,$$

oder es ist

$$\Sigma_x (-)^{x-1} \left\| \begin{matrix} a_1 \ldots\ldots\ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n\, B_x \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots b_m \\ B_1 \ldots B_{x-1}\, B_{x+1} \ldots B_{m+1} \end{matrix} \right)$$

$$= \Sigma_y (-)^{n-y+1} \left\| \begin{matrix} a_1 \ldots a_{y-1}\, a_{y+1} \ldots a_{n+1} \\ A_1 \ldots\ldots\ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_y\, b_1\, b_2 \ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right)$$

wo $x = 1, 2, \ldots\ldots, m+1,$

$y = 1, 2, \ldots\ldots, n+1;$

oder es ist

$$\left.\begin{matrix} \left\| \begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_2 \ldots\ldots B_{m+1} \end{matrix} \right) \\ - \left\| \begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_2 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_1\, B_3 \ldots\ldots B_{m+1} \end{matrix} \right) \\ + \left\| \begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_3 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_1\, B_2\, B_4 \ldots B_{m+1} \end{matrix} \right) \\ - \ldots\ldots\ldots\ldots\ldots\ldots \\ + (-)^m \left\| \begin{matrix} a_1 \ldots\ldots a_{n+1} \\ A_1 \ldots A_n\, B_{m+1} \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots b_m \\ B_1 \ldots\ldots B_m \end{matrix} \right) , \end{matrix} \right\}$$

$$= \left.\begin{matrix} \left\| \begin{matrix} a_1 \ldots\ldots a_n \\ A_1 \ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_{n+1}\, b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right) \\ - \left\| \begin{matrix} a_1 \ldots a_{n-1}\, a_{n+1} \\ A_1 \ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_n\, b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right) \\ + \left\| \begin{matrix} a_1 \ldots a_{n-2}\, a_n\, a_{n+1} \\ A_1 \ldots\ldots A_n \end{matrix} \right) \cdot \left\| \begin{matrix} a_{n-1} b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right) \\ - \ldots\ldots\ldots\ldots\ldots\ldots \\ (-)^n \left\| \begin{matrix} a_2 \ldots\ldots a_{n+1} \\ A_1 \ldots\ldots A_n \end{matrix} \right) . \left\| \begin{matrix} a_1\, b_1 \ldots\ldots b_m \\ B_1 \ldots\ldots B_{m+1} \end{matrix} \right) \end{matrix} \right\} . \qquad \text{(L.)}$$

The further generalisation of which this is possible, and which Schweins effects, depends on the fact, that the law of formation twice used in proving the identity, is but the simplest case of Laplace's expansion-theorem, and that the said theorem can be similarly used in all its generality. In other words, instead of taking only *one* of the B's at a time to go along with the A's to form the first factors of the left-hand aggregate, we may take any fixed number of them. For example, out of six B's we may take every set of *three* to go along with two A's, and we shall have the aggregate

$$
\begin{aligned}
&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_3}\right)\cdot\left\Vert \overset{b_1}{B_4}\overset{b_2}{B_5}\overset{b_3}{B_6}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_4}\right)\cdot\left\Vert \overset{b_1}{B_3}\overset{b_2}{B_5}\overset{b_3}{B_6}\right)\\
+&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_5}\right)\cdot\left\Vert \overset{b_1}{B_3}\overset{b_2}{B_4}\overset{b_3}{B_6}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_3}\overset{b_2}{B_4}\overset{b_3}{B_5}\right)\\
+&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_3}\overset{a_5}{B_4}\right)\cdot\left\Vert \overset{b_1}{B_2}\overset{b_2}{B_5}\overset{b_3}{B_6}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_3}\overset{a_5}{B_5}\right)\cdot\left\Vert \overset{b_1}{B_2}\overset{b_2}{B_4}\overset{b_3}{B_6}\right)\\
+&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_3}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_2}\overset{b_2}{B_4}\overset{b_3}{B_5}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_4}\overset{a_5}{B_5}\right)\cdot\left\Vert \overset{b_1}{B_2}\overset{b_2}{B_3}\overset{b_3}{B_6}\right)\\
+&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_4}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_2}\overset{b_2}{B_3}\overset{b_3}{B_5}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_5}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_2}\overset{b_2}{B_3}\overset{b_3}{B_4}\right)\\
+&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_3}\overset{a_5}{B_4}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_5}\overset{b_3}{B_6}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_3}\overset{a_5}{B_5}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_4}\overset{b_3}{B_6}\right)\\
+&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_3}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_4}\overset{b_3}{B_5}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_4}\overset{a_5}{B_5}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_3}\overset{b_3}{B_6}\right)\\
+&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_4}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_3}\overset{b_3}{B_5}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_2}\overset{a_4}{B_5}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_3}\overset{b_3}{B_4}\right)\\
+&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_3}\overset{a_4}{B_4}\overset{a_5}{B_5}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_6}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_3}\overset{a_4}{B_4}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_5}\right)\\
+&\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_3}\overset{a_4}{B_5}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_4}\right) - \left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_4}\overset{a_4}{B_5}\overset{a_5}{B_6}\right)\cdot\left\Vert \overset{b_1}{B_1}\overset{b_2}{B_2}\overset{b_3}{B_3}\right),
\end{aligned}
$$

—the sign of any term being determined by the number of inversions of order among the suffixes of all the B's of the term. In this particular case the first use of Laplace's expansion-theorem is to transform

$$\left\Vert \overset{a_1}{A_1}\overset{a_2}{A_2}\overset{a_3}{B_1}\overset{a_4}{B_2}\overset{a_5}{B_3}\right)$$

and the other similar determinants each into an aggregate of ten products, the two factors of any product in the expansion of

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ A_1 & A_2 & B_1 & B_2 & B_3 \end{matrix} \right)$$

being, as we should nowadays say, a minor formed from the first two rows and the complementary minor. In this way would arise 20 rows of 10 terms each, and these being combined by a second use of Laplace's expansion-theorem in columns of 20 terms each, the outcome would be an aggregate of 10 products, viz., the aggregate

$$\left\| \begin{matrix} a_1 & a_2 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_3 & a_4 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_3 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_2 & a_4 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 & a_4 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_2 & a_3 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_1 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_2 & a_3 & a_4 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_2 & a_3 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_4 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_2 & a_4 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_3 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_2 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_3 & a_4 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_3 & a_4 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_2 & a_5 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_3 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_2 & a_4 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right) - \left\| \begin{matrix} a_4 & a_5 \\ A_1 & A_2 \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{matrix} \right).$$

The following is Schweins' statement of this most general theorem:— (L. 2)

$$\Sigma(-)^{*} \left\| \begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a_n \\ A_1 \ldots . A_{n-q}\, B'_1 \ldots\ldots B' \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots b_{m-q} \\ B'_{q+1} \ldots\ldots B'_m \end{matrix} \right)$$

$$= \Sigma(-)^{*} \left\| \begin{matrix} a'_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots a'_{n-q} \\ A_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots A_{n-q} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{n-q+1} \ldots a'_n\, b_1 \ldots b_{m-q} \\ B_1 \ldots\ldots\ldots\ldots B_m \end{matrix} \right).$$

The only points about it requiring explanation are the exact effect to be given to the symbol Σ, and the meaning of the dashes affixed to certain of the letters. The two symbols are connected with each other, the dashes not being permanently attached to the letters, but merely put in to assist in explaining the duty of the Σ. On the left-hand member of the identity, the two symbols indicate that the first term is got by dropping the dashes, and that from this first term another term is got, if we substitute for $B_1 \ldots . B_q$, some other set of q B's chosen from $B_1 \ldots . B_m$, and take the remaining B's to form the B's of the second determinant,—the two sets of

B's being in both cases first arranged in ascending order of their suffixes. On the other side of the identity, the use of the symbols is exactly similar, $n-q$ of the n upper elements $a_1, \ldots\ldots, a_n$ being taken for the first determinant of any term of the series, and the remainder for the second determinant. The number of terms in the series on the one side is evidently $m!/q!(m-q)!$ and on the other $n!/q!(n-q)!$

In the demonstration of the theorem greater fulness is evidently necessary than in the case of the previous theorem, the rule of signs in particular requiring attention. This Schweins' does not give. He merely tells the character of the first transformation, symbolising the expansion obtainable, and then says that a recombination is possible, giving the result.

The succeeding five pages (pp. 350–355) are devoted to evolving and stating special cases. This is by no means unnecessary work, as in the case of a theorem of so great generality it is often a matter of some trouble to ascertain whether a particular given result be really included in it or not. To students of the history of the subject the special cases are doubly interesting, because it is in them we may expect to find links of connection with the work of previous investigators.

The first descent from generality is made by putting some of the B's equal to A's, the theorem then being (L. 3)

$$\Sigma(-)^{*}\left\| \begin{matrix} a_1 \ldots\ldots\ldots\ldots\ldots\ldots a_{p+s+q} \\ A_1 \ldots A_{p+s} B'_1 \ldots\ldots B'_q \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots\ldots b_{k+p} \\ B'_{q+1} \ldots B_{q+k} A_1 \ldots\ldots A_p \end{matrix} \right)$$

$$= \Sigma(-)^{*}\left\| \begin{matrix} a'_1 \ldots\ldots a'_{p+s} \\ A_1 \ldots\ldots A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s+1} \ldots\ldots a'_{p+s+q}\, b_1 \ldots\ldots\ldots b_{k+p} \\ B_1 \ldots\ldots\ldots\ldots B_{q+k} A_1 \ldots\ldots A_p \end{matrix} \right).$$

If in addition to this specialisation, some of the b's be put equal to the a's, the result is (L. 4)

$$\Sigma(-)^{*}\left\| \begin{matrix} b_1 \ldots b_h\, a'_1 \ldots\ldots a'_{p+s-h} \\ A_1 \ldots\ldots\ldots\ldots A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s-h+1} \ldots\ldots a'_{p+s-h+q},\, b_1 \ldots b_{h+k} \\ B_1 \ldots\ldots B_{h+k-p+q} A_1 \ldots\ldots\ldots A_p \end{matrix} \right)$$

$$= \Sigma(-)^{*}\left\| \begin{matrix} b_1 \ldots\ldots b_h\, a_1 \ldots\ldots\ldots a_{p+s+q-h} \\ A_1 \ldots\ldots A_{p+s} B'_1 \ldots B'_q \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \ldots\ldots\ldots\ldots\ldots\ldots b_{h+k} \\ B'_{q+1} \ldots B'_{q+h+k-p} A_1 \ldots A_p \end{matrix} \right)$$

—a notable theorem, which it would not be inappropriate to consider rather as a generalisation than as a special case of the theorem from which it is derived. Returning, however, to the preceding case, and putting $k=0$, we obtain (L. 5)

$$\left\| \begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots a_{p+s+q} \\ A_1 \ldots A_{p+s}\, B_1 \ldots B_q \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \cdots\cdots\cdots b_p \\ A_1 \ldots\ldots A_p \end{matrix} \right)$$

$$= \Sigma(-)^* \left\| \begin{matrix} a'_1 \cdots\cdots a'_{p+s} \\ A_1 \ldots\ldots A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s+1} \cdots\cdots a'_{p+s+q}\, b_1 \cdots\cdots b_p \\ B_1 \ldots\ldots B_q\, A_1 \ldots\ldots\ldots A_p \end{matrix} \right) .$$

This may be viewed as an extension of Laplace's expansion-theorem to which it degenerates when p is put equal to 0. Though a comparatively very special identity it is considerably beyond anything attained by Schweins' predecessors. In fact, we only come upon something like known ground, when in descending further, we put in it $q=1$. The result thus obtained is

$$\left\| \begin{matrix} a_1 \cdots\cdots\cdots\cdots a_{p+s+1} \\ A_1 \ldots\ldots A_{p+s}\, B_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \cdots\cdots\cdots\cdots b_p \\ A_1 \ldots\ldots\ldots A_p \end{matrix} \right)$$

$$= \Sigma(-)^* \left\| \begin{matrix} a'_1 \cdots\cdots\cdots\cdots a'_{p+s} \\ A_1 \ldots\ldots\ldots A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s+1}\, b_1 \cdots\cdots b_p \\ B_1 \quad A_1 \ldots\ldots A_p \end{matrix} \right), \quad (\text{XLVI. } 2).$$

which closely resembles a theorem of Desnanot's. The difference between them consists in the fact that here the second factor on the left-hand side is *any* determinant of a lower order than the cofactor, whereas in Desnanot the second factor is a *minor* of the cofactor. A further specialisation, viz. putting $B_1 = A_{p+1}$, brings us to the result

$$\left.\begin{aligned} &\Sigma(-)^* \left\| \begin{matrix} a'_1 \cdots\cdots a'_{p+s} \\ A_1 \ldots\ldots A_{p+s} \end{matrix} \right) \cdot \left\| \begin{matrix} a'_{p+s+1},\, b_1 \cdots\cdots b_p \\ A_1 \ldots\ldots\ldots A_{p+1} \end{matrix} \right) = 0 , \\ &\text{or} \\ &\Sigma(-)^* \left\| \begin{matrix} b_1 \cdots\cdots\cdots b_{p+1} \\ A_1 \ldots\ldots A_p\, B_1 \end{matrix} \right) \cdot \left| \begin{matrix} b_1 \quad b_2 \cdots\cdots b_{p+2} \\ B'_2,\, B'_3 \ldots\ldots B'_{p+3} \end{matrix} \right) = 0 . \end{aligned}\right\} \quad (\text{XXIII. } 7).$$

The form here is that of a vanishing aggregate of products of pairs of determinants, and identities of this form we have already had to consider in dealing with Bézout, Monge, Cauchy, and Desnanot. To the last of these only does Schweins refer. His words are (p. 352)—

> "Wird in dieser Gleichung $s=2$ gesetzt, so entsteht folgende:—
>
> $$\Sigma(-)^* \left\| \begin{matrix} b_1 \cdots\cdots\cdots b_{p+1} \\ A_1 \ldots A_p\, B'_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 \quad b_2 \cdots\cdots b_{p+2} \\ B'_2\, B'_3 \ldots\ldots B'_{p+3} \end{matrix} \right) = 0 ,$$
>
> wovon Desnanot einige ganz specielle Fälle gefunden hat, oder vielmehr der ganze Inhalt seiner Untersuchung ist in folgenden dreien Gleichungen begriffen

$$\Sigma(-)^{*}\left\| \begin{matrix} b_1 & b_2 & b_3 \\ A_1 & A_2 & B'_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 & b_4 \\ B'_2 & B'_3 & B'_4 & B'_5 \end{matrix} \right) = 0 ,$$

$$\Sigma(-)^{*}\left\| \begin{matrix} b_1 & b_2 \\ A_1 & B'_1 \end{matrix} \right) \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 \\ B'_2 & B'_3 & B'_4 \end{matrix} \right) = 0 ,$$

$$\Sigma(-)^{*} \begin{matrix} b_1 \\ B'_1 \end{matrix} \cdot \left\| \begin{matrix} b_1 & b_2 & b_3 & b_4 \\ B'_2 & B'_3 & B'_4 & B'_5 \end{matrix} \right) = 0 ,$$

welche mit ermüdender Weitläufigkeit bewiesen sind . . . . . ."

This statement is unfortunately not by any means accurate. As for the "ermüdende Weitläufigkeit," there can be no doubt about it, and to assert its existence is fair criticism; but to say that the whole of Desnanot's results are to be found in the three identities specified is a misrepresentation of the actual facts, and therefore quite unfair. The reader has only to turn back for a moment to our account of Desnanot's work, to verify the fact that the two most important general results attained by the latter (XXIII. 6 and XLVI.) are ignored by Schweins altogether.

The remaining paragraphs of the chapter are taken up with the very elementary case in which the products are three in number, and the theorem itself nothing more than one of the extensionals so lengthily dwelt upon by Desnanot, viz., the extensional

$$a_1|b_1c_2| \;-\; b_1|a_1c_2| \;+\; c_1|a_1b_2| \;=\; 0 .$$

It is written in several forms, *e.g.*—

$$\left\| \begin{matrix} a_1 \cdots\cdots\cdots\cdots a_{n+m} \\ A_1 \;.\;.\;.\;.\;.\;.\;.\;.\;.\; A_{n+m} \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots a_{n+m+1} \\ A_1 \ldots A_{n-1}A_{n+1} \ldots A_{n+m+1}B \end{matrix} \right)$$

$$- \left\| \begin{matrix} a_1 \cdots\cdots\cdots\cdots a_{n+m} \\ A_1 \ldots A_{n-1}A_{n+1} \ldots A_{n+m+1} \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots a_{n+m+1} \\ A_1 \;.\;.\;.\;.\;.\;.\;.\;.\; A_{n+m} \;\; B \end{matrix} \right)$$

$$+ \left\| \begin{matrix} a_1 \cdots\cdots\cdots\cdots a_{n+m} \\ A_1 \ldots A_{n-1}A_{n+1} \ldots A_{n+m}B \end{matrix} \right) \cdot \left\| \begin{matrix} a_1 \cdots\cdots\cdots\cdots\cdots a_{n+m+1} \\ A_1 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; A_{n+m+1} \end{matrix} \right) = 0 .$$

The next chapter, the third, concerns the solution of a set of linear equations, although according to the title its subject is the transformation of determinants into other determinants when the elements are connected by linear equations. It presents no new feature.

The fourth chapter deals with a special form of determinant, the consideration of which must therefore be deferred. Suffice it for the present to say, as an evidence of Schweins' grasp of the subject,

that the solution of the problem attempted is complete and the result very interesting.

The fifth gives the solution of a problem on which the general Theory of Series is said to depend, the problem being the transformation of

$$\frac{\left\| \begin{matrix} \mathrm{a}_1 & \mathrm{a}_2 & \cdots & & \cdots & \mathrm{a}_\infty \\ \mathrm{B} & \mathrm{A} & . \ . & \mathrm{A}_{n-1}\mathrm{A}_{n+1} & . \ . \ . & \mathrm{A}_\infty \end{matrix} \right)}{\left\| \begin{matrix} \mathrm{a}_1 & \mathrm{a}_2 & \cdots & \mathrm{a}_\infty \\ \mathrm{A}_1 & \mathrm{A}_2 & \cdots & \mathrm{A}_\infty \end{matrix} \right)}$$

into an unending series. The numerator, it will be observed, is of the order ∞ : the denominator is of the same order: and all the rows of the former except one occur in the latter. Indeed, if the first row of the numerator were deleted, and the n^{th} row of the denominator, there would be nothing to distinguish the one from the other. The subject is best illustrated by a special example in more modern notation. Recurring to the extensional above referred to as the concluding theorem of the second chapter, and taking the case where the factors are of the 4^{th} and 5^{th} orders, we manifestly have

$$|a_1b_2c_3e_4|.|a_1b_2c_3d_4f_5| - |a_1b_2c_3d_4|.|a_1b_2c_3e_4f_5| + |a_1b_2c_3f_4|.|a_1b_2c_3e_4d_5| = 0\,,$$

from which, on dividing by $|a_1b_2c_3e_4|.|a_1b_2c_3e_4d_5|$, we obtain

$$\frac{|a_1b_2c_3d_4f_5|}{|a_1b_2c_3e_4d_5|} - \frac{|a_1b_2c_3d_4|.|a_1b_2c_3e_4f_5|}{|a_1b_2c_3e_4|\,|a_1b_2c_3e_4d_5|} + \frac{|a_1b_2c_3f_4|}{|a_1b_2c_3e_4|} = 0\,.$$

Similarly

$$\frac{|a_1b_2c_3f_4|}{|a_1b_2e_3c_4|} - \frac{|a_1b_2c_3|.|a_1b_2e_3f_4|}{|a_1b_2d_3|\,|a_1b_2e_3c_4|} + \frac{|a_1b_2f_3|}{|a_1b_2e_3|} = 0\,,$$

$$\frac{|a_1b_2f_3|}{|a_1e_2b_3|} - \frac{|a_1b_2|.|a_1e_2f_3|}{|a_1e_2|\,|a_1e_2b_3|} + \frac{|a_1f_2|}{|a_1e_2|} = 0\,,$$

and
$$\frac{|a_1f_2|}{|e_1a_2|} - \frac{a_1}{e_2}.\frac{|e_1f_2|}{|e_1a_2|} + \frac{f_1}{d_1} = 0\,,$$

the last fraction of each identity, be it observed, being the same as the first of the next with its sign changed. From the four by addition we have

$$\frac{|a_1b_2c_3d_4f_5|}{|a_1b_2c_3d_4e_5|} = \frac{|a_1b_2c_3d_4|.|a_1b_2c_3e_4f_5|}{|a_1b_2c_3e_4|\,|a_1b_2c_3d_4e_5|}$$

$$+ \frac{|a_1b_2c_3|.|a_1b_2e_3f_4|}{|a_1b_2e_3|\,|a_1b_2c_3e_4|}$$

$$+ \frac{|a_1b_2|.|a_1e_2f_3|}{|a_1e_2|\,|a_1b_2e_3|}$$

$$+ \frac{a_1}{e_1}.\frac{|e_1f_2|}{|a_1e_2|}$$

$$+ \frac{f_1}{e_1}.$$

The general result, as stated by Schweins, is that

$$(-)^{n+1}\frac{\left\|\begin{matrix} a_1 & \cdots & & \cdots & a_{n+m+1} \\ \mathrm{B}\ \mathrm{A}_1 & \cdots & \mathrm{A}_{n-1}\mathrm{A}_{n+1} & \cdots & \mathrm{A}_{n+m+1} \end{matrix}\right)}{\left\|\begin{matrix} a_1 & \cdots & a_{n+m+1} \\ \mathrm{A}_1 & \cdots & \mathrm{A}_{n+m+1} \end{matrix}\right)}$$

$$= \mathrm{L}^{(n)}_0 . \mathrm{V}^{(n)} - \mathrm{L}^{(n)}_1 . \mathrm{V}^{(n+1)} + \mathrm{L}^{(n)}_2 . \mathrm{V}^{(n+2)} - \ldots . (-)^{m+1}\mathrm{L}^{(n)}_{m+1} . \mathrm{V}^{(n+m+1)},$$

$$\text{where}\quad \mathrm{L}^{(n)}_m = \frac{\left\|\begin{matrix} a_1 & \cdots & & \cdots & a_{n+m-1} \\ \mathrm{A}_1 & \cdots & \mathrm{A}_{n-1}\mathrm{A}_{n+} & \cdots & \mathrm{A}_{n+m} \end{matrix}\right)}{\left\|\begin{matrix} a_1 & \cdots & a_{n+m-1} \\ \mathrm{A}_1 & \cdots & \mathrm{A}_{n+m-1} \end{matrix}\right)},$$

$$\text{and}\quad \mathrm{V}^{(n+m)} = \frac{\left\|\begin{matrix} a_1 & \cdots & & a_{n+m} \\ \mathrm{A}_1 & \cdots & \mathrm{A}_{n+m-1} & \mathrm{B} \end{matrix}\right)}{\left\|\begin{matrix} a_1 & \cdots & a_{n+m} \\ \mathrm{A}_1 & \cdots & \mathrm{A} \end{matrix}\right)}. \qquad \text{(LI.)}$$

Since the expression thus expanded is itself one of the L's, viz., $\mathrm{L}^{(n)}_{m+2}$—as is readily seen by transferring the B from the beginning to the end, and denoting it by A_{n+m+2},—and since $\mathrm{L}^{(n)}_0 = 1$, the identity may equally appropriately be written with $\mathrm{L}^{(n)}_{m+2}$ at the end of the right-hand member, and looked upon as the recurring law of formation of the L's in terms of the V's. This Schweins does, giving indeed the result of solving for $\mathrm{L}^{(n)}_1$, $\mathrm{L}^{(n)}_2$, . . .

The Second Section, consisting of five chapters, and extending to 30 pp., is devoted to a special form of determinants, viz., those already partly investigated by Cauchy, and afterwards known as alternants.

The Third Section, extending only to 4 pp., deals with another special form, whose elements are finite differences of a set of functions.

The Fourth Section, consisting of four chapters, and extending to 27 pp., has for its subject a third special form, foreshadowed by Wronski, the characteristic of which is that one of the indices denotes repetition of an operation involving differentiation.

When these Sections come to be considered in their proper places, it will be seen that very great credit is due to Schweins for his labours, and that he has been most undeservedly neglected. The fact that he had ever written on determinants was only brought to light in 1884: * and, so far as can be gathered, his treatise had no influence whatever either on the work of succeeding investigators, or in diffusing a knowledge of the subject.

JACOBI (1827).

[Ueber die Hauptaxen der Flächen der zweiten Ordnung. *Crelle's Journal*, ii. pp. 227–233.]

[De singulari quadam duplicis Integralis transformatione. *Crelle's Journal*, ii. pp. 234–242.]

[Ueber die Pfaffsche Methode, eine gewöhnliche lineäre Differentialgleichung zwischen $2n$ Variabeln durch ein System von n Gleichungen zu integriren. *Crelle's Journal*, ii. pp. 347–357.]

We come here simultaneously on the names of a great mathematician and a great mathematical journal. Crelle's *Journal für die reine und angewandte Mathematik*, which began to appear at the end of the year 1825, and which without any of the symptoms of old age still survives, has rendered on more than one occasion important service towards the advancement of the theory of determinants. Its first contributor on the subject and one of its greatest was Jacobi. At a later date he published in the *Journal* an excellent monograph on Determinants; but even his earliest papers show that he had begun to find it a useful weapon of research.

In the first of the memoirs above noted, dealing with the subject

* v. *Phil. Mag.* for November: *An overlooked Discoverer in the Theory of Determinants.*

of orthogonal substitution, constant use is, of course, made of the functions; but there is no special notation employed, nor indeed anything to indicate that the expressions used were members of a class having properties peculiar to themselves.

In the second memoir, which likewise is taken up with a transformation, but in which the sets of equations involve *four* unknowns, any special notation is still avoided. Expressions, readily seen to be determinants of the third order, are even not set down, because, as the author expressly states, they would be too lengthy. The last clause of the passage in which this statement occurs is noteworthy. The words are (p. 236)—

"Dato systemate æquationum

$$\begin{aligned}
\alpha\ u + \beta\ x + \gamma\ y + \delta\ z &= m, \\
\alpha'\ u + \beta'\ x + \gamma'\ y + \delta'\ z &= m, \\
\alpha''u + \beta''x + \gamma''y + \delta''z &= m'', \\
\alpha'''u + \beta'''x + \gamma'''y + \delta'''z &= m''',
\end{aligned}$$

"ponamus earum resolutione erui:

$$\begin{aligned}
\mathrm{A}m + \mathrm{A}'m' + \mathrm{A}''m'' + \mathrm{A}'''m''' &= u, \\
\mathrm{B}m + \mathrm{B}'m' + \mathrm{B}''m'' + \mathrm{B}'''m''' &= x, \\
\mathrm{C}m + \mathrm{C}'m' + \mathrm{C}''m'' + \mathrm{C}'''m''' &= y, \\
\mathrm{D}m + \mathrm{D}'m' + \mathrm{D}''m'' + \mathrm{D}'''m''' &= z.
\end{aligned}$$

"Valores sedecim quantitatum A, B, etc., supprimimus eorum prolixitatis causa; in libris algebraicis passim traduntur, et algorithmus, cuius ope formantur, hodie abunde notus est."

On the next page, in eliminating D, D′, D″, D‴ from the set of equations

$$\begin{aligned}
0 &= \mathrm{D}(a-x) + \mathrm{D}'b' + \mathrm{D}''b'' + \mathrm{D}'''b''', \\
0 &= \mathrm{D}b' + \mathrm{D}'(a'+x) + \mathrm{D}''c''' + \mathrm{D}'''c'', \\
0 &= \mathrm{D}b'' + \mathrm{D}'c''' + \mathrm{D}''(a''+x) + \mathrm{D}'''c', \\
0 &= \mathrm{D}b''' + \mathrm{D}'c'' + \mathrm{D}''c' + \mathrm{D}'''(a'''+x),
\end{aligned}$$

he arranges the resultant as one would now do who had expanded it from the determinant form according to products of the elements of the principal diagonal, viz., he says (p. 238)—

"Fit illa, eliminationis negotio rite instituto

$$
\begin{aligned}
0 = \quad & (a-x)(a'+x)(a''+x)(a'''+x) \\
& -(a-x)(a'+x)c'c' - (a-x)(a''+x)c''c'' - (a-x)(a'''+x)c'''c''' \\
& -(a''+x)(a'''+x)b'b' - (a'''+x)(a'+x)b''b'' - (a'+x)(a''+x)b'''b''' \\
& +2c'c''c'''(a-x) + 2c'b''b'''(a'+x) \qquad\qquad (\text{LII.}) \\
& +2c''b'''b'(a''+x) + 2c'''b'b''(a'''+x) \\
& +b'b'c'c' + b''b''c''c'' + b'''b'''c'''c''' - 2b'b''c'c'' - 2b''b'''c''c''' - 2b'''b'c'''c'.
\end{aligned}
$$

From the next paragraph we learn his sources of information, and infer that as yet Cauchy's memoir was unknown to him. The first sentence is (p. 239)—

> "Inter sedecim quantitates α, β, etc. et sedecim, quæ ex iis derivantur, A, A′, etc. plurimæ intercedunt relationes perelegantes, quæ cum analystis ex iis, quæ Laplace, Vandermonde, in commentariis academiæ Parisiensis A. 1772 p. ii., Gauss in disquis. arithm. sectio V., J. Binet in vol. ix. diariorum instituti polytechnici Pariensis, aliique tradiderunt, satis notæ sint, paucas tantum referam, quæ casu nostro speciali ope æquationum (IV) facile ex iis derivantur."

The third memoir is by far the most important to us. In the course of the investigation a new special form of determinants, afterwards so well known by the designation *skew* determinants, turns up; and three pages are devoted to an examination of the final expanded form of it. This examination, we cannot, of course, now enter upon; but it is of importance to note that in it Jacobi takes the step of adopting the name *determinant*,—a fact which doubtless was decisive of the fate of the word. The adoption thus made (although stated to be from Gauss), and the occurrence of the words "Horizontalreihen," "Verticalreihen," make it probable that Cauchy's memoir had now come to his notice.

REISS (1829).

[Mémoire sur les fonctions semblables de plusieurs groupes d'un certain nombre de fonctions ou élémens. *Correspondance math. et phys.*, v. pp. 201–215.]

In Reiss we have an author who starts to his subject as if it were entirely new, the only preceding mathematician whom he

mentions being Lagrange. Like Cauchy he opens by explaining a mode of forming functions more general than those of which he afterwards treats, the essence of it being that an expression involving several of the $n.\nu$ quantities,

$$\begin{array}{ccccc} a^{\alpha} & a^{\beta} & a^{\gamma} & \ldots & a^{\rho} \\ b^{\alpha} & b^{\beta} & b^{\gamma} & \ldots & b^{\rho} \\ c^{\alpha} & c^{\beta} & c^{\gamma} & \ldots & c^{\rho} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r^{\alpha} & r^{\beta} & r^{\gamma} & \ldots & r^{\rho} \end{array}$$

is taken, and each *exponent* ("exposant") changed successively with all the other exponents, $\alpha, \beta, \ldots$, or each *base* changed with all the other bases, $a, b, \ldots$. Only a line or two, however, is given to this, the special class known to us as determinants being taken up at once.

His notation for

$$a^1b^2c^3 - a^1b^3c^2 - a^2b^1c^3 + a^2b^3c^1 + a^3b^1c^2 - a^3b^2c^1$$

is

$$(abc\,,\,\overline{123})\,, \qquad \text{(VII. 7)}$$

a line being drawn above the exponents to indicate permutation. His rule of formation of the terms and rule of signs are combined after the manner of Hindenburg. Like Hindenburg, he arranges the permutations as one arranges numbers in increasing order of magnitude; but, unlike Hindenburg, after the arrangement has been made he determines the sign of any *particular* term. On this point his words are (p. 202)

> "Cela fait, déterminons généralement le signe du M^{me} produit (soit $\dot{M}$) de la manière suivante. Le nombre M sera renfermé entre les produits $1.2.3\ldots l$ et $1.2.3\ldots l(l+1)$; soit $M = m + \lambda \times 1.2.3\ldots l$, de sorte que $\lambda < l+1$, et $m > 0$ et $< 1 + 1.2.3\ldots l$. Cela étant, faisons $\dot{M} = \dot{m}(-1)^{\lambda}$." (III. 24.)

This apparently means that if the sign of the 23rd term in the expansion of

$$(abcd\,,\,\overline{1234})^*$$

be wanted, we divide 23 by 1.2.3, getting the quotient 3 and the remainder 5, and thence conclude that the sign wanted is got from

* Or $(abcde\,,\overline{12345})$, or indeed $(a_1a_2\ldots a_n,\,\overline{123\ldots n})$.

the sign of the 5^{th} term by multiplying the latter by $(-1)^3$. Of course 5 has then to be dealt with after the manner of 23, the quotient and remainder this time being 2 and 1, so that we conclude that the sign of the 5^{th} term is got from the sign of the 1^{st} term by multiplying by $(-1)^2$. And the sign of the 1^{st} term being $+$, the sign of the 23^{rd} is thus seen to be

$$(-1)^{3+2} \quad i.e. \quad -.$$

It would seem at first as if the case where M is itself a factorial were neglected. This however is not so, the condition $m < 1 + 1.2.3 \ldots l$ being corrective of the opening statement that M must lie between $1.2.3 \ldots l$ and $1.2.3 \ldots l\,(l+1)$. For example, the term being the 24^{th}, we put 24 in the form $3 \times 1.2.3 + 6$, and thus learn that the sign required is different from the sign of the 6^{th} term: then we put 6 in the form $2 \times 1.2 + 2$, and thus learn that the sign of the 6^{th} term is the same as the sign of the 2^{nd} term; finally, we put 2 in the form $1 \times 1 + 1$, which shows that the sign of the 2^{nd} term differs from the sign of the 1^{st}: the conclusion of the whole being that the signs of the 24^{th} and 1^{st} terms are the same, or that they are connected by the factor $(-1)^{3+2+1}$.

Though interesting in itself, a more troublesome form of the rule of signs for the purposes of demonstration it is scarcely possible to conceive, and, as might therefore be expected, it is on the score of logical development that Reiss' paper is weak. Through inability to use the rule later in the demonstration of the so-called Laplace's expansion-theorem, he is forced to supplement it by another convention. His words are (p. 203)—

"Avant d'aller plus loin, faisons encore la détermination suivante. Soit ω une fonction quelconque dans laquelle les k quantités $A, B, C, \ldots A^k$ entrent d'une manière quelconque. Supposons que ces dernières soient les k premières de l'échelle $\begin{pmatrix} A & B & C & \ldots & A^k & \ldots & S \\ 1 & 2 & 3 & \ldots & k & \ldots & s \end{pmatrix}$. Qu'on fasse avec ces s élémens toutes les combinaisons sans répétition de la classe k, et qu'on les substitue successivement au lieu de $A, B, \ldots A^k$ dans la fonction ω; c'est-à-dire le premier élément de chaque

combinaison à A, le second à B, etc. Nous obtiendrons par là autant de fonctions semblables à ω qu'il y a de combinaisons de la classe k de s élémens. Or, entre toutes les combinaisons qui en précèdent une quelconque, il s'en trouvera une qui aura $k-1$ élémens communs avec elle, tandis que les deux élémens qui restent isolés dans l'une et l'autre se suivent immédiatement dans l'échelle. Donnons à la fonction qui contient la dernière de ces combinaisons le signe opposé à celui de l'autre fonction; par conséquent les signes de toutes les fonctions semblables à ω seront parfaitement déterminés, et dépendront du signe de la première fonction (f (A,B,C, . . . A^k)). Soit, par exemple, $s=5$, $k=3$; nous aurons successivement, en remplaçant A,B,C, . . . S par 1, 2, 3, 4, 5, et en donnant le signe (+) à f (123),

$$\begin{array}{l} +f(123), -f(124), +f(125), +f(134), -f(135) \\ +f(145), -f(234), +f(235), -f(245), +f(345). \end{array}$$

Voici comment on déterminera le signe de chaque fonction semblable à ω d'après celui d'une antre quelconque. Qu'on cherche les nombres qui se trouvent dans l'échelle $\begin{pmatrix} A & B & C & . . . & A^k & . . . & S \\ 1 & 2 & 3 & . . . & k & . . . & s \end{pmatrix}$ sous les élémens de l'une et de l'autre de ces fonctions. Si l'on nomme h et h' leurs sommes respectives, on trouvera le signe de l'une des fonctions $=$ $(-1)^{h'-h} \times$ le signe de l'autre."

Four theorems he considers fundamental, viz., those known to us as (1) Bézout's recurrent law of formation, in all its generality; (2) Vandermonde's proposition that permutation of bases leads to the same result as permutation of exponents; (3) Laplace's expansion-theorem; (4) Vandermonde's proposition regarding the effect of making two bases or two exponents equal. The two most important, viz. (1) and (2), he leaves without proof, and the 4th he says he would at once deduce from the 3rd,—doubtless by choosing the expansion in which the first factor of every term would be of the form

$$(a\alpha, \overline{a\beta})$$

and therefore equal to zero.

The proof of the 2[nd] theorem, viz.,

$$(abc\ldots r, \overline{\alpha\beta\gamma\ldots\rho}) = \overline{(abc\ldots r}, \alpha\beta\gamma\ldots\rho),$$

is by the method of so-called induction, and may be illustrated in a later notation by considering the case

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

From theorem (1) we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix},$$

$$= -b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix},$$

$$= c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

But by hypothesis all the determinants on the right here may have their rows changed into columns; and this being done we have by addition and the use of theorem (1)—

$$3\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and thence the identity required. (IX. 4)

To this proof the following note is appended (p. 207) :—

"Cette démonstration quoiqu'assez simple semble reposer cependant sur un artifice de calcul: mais en cherchant une démonstration *directe*, j'ai rencontré une difficulté d'un genre particulier. En effet, on trouve facilement que l^{me} terme de l'une des fonctions en question est aussi égal ou au même terme de l'autre, ou généralement au m^{me}, et que, dans le

combinaison à A, le second à B, etc. Nous obtiendrons par là autant de fonctions semblables à ω qu'il y a de combinaisons de la classe k de s élémens. Or, entre toutes les combinaisons qui en précèdent une quelconque, il s'en trouvera une qui aura $k-1$ élémens communs avec elle, tandis que les deux élémens qui restent isolés dans l'une et l'autre se suivent immédiatement dans l'échelle. Donnons à la fonction qui contient la dernière de ces combinaisons le signe opposé à celui de l'autre fonction; par conséquent les signes de toutes les fonctions semblables à ω seront parfaitement déterminés, et dépendront du signe de la première fonction $(f(\mathrm{A,B,C}, \ldots \mathrm{A}^k))$. Soit, par exemple, $s=5$, $k=3$; nous aurons successivement, en remplaçant $\mathrm{A,B,C}, \ldots \mathrm{S}$ par 1, 2, 3, 4, 5, et en donnant le signe $(+)$ à $f(123)$,

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$$= c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

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dernier cas, le m^{me} terme de la première est aussi égal au l^{me} de la seconde, abstraction faite des signes. (IX. 5) Mais l'identité de ces derniers (qui est de rigueur) exige des explications très-longues et beaucoup moins élémentaires que la démonstration que je viens de donner."

The remaining six or seven pages of the paper are more interesting, and concern the subject of vanishing aggregates of products of pairs of determinants. The theorems were suggested by taking, as we now say, a determinant of even order having its last n rows identical with its first n rows, *e.g.*, the determinant

$$(abab, \overline{1234}),$$

and using theorem (3) to expand it in terms of minors formed from the first n rows and their complementary minors. When n is even, a proof is thus obtained, as we have seen in the footnote to the account of Bézout's paper of 1779, that the first half of the expansion is equal to zero. When n is odd, the method fails, although the proposition is still true.* Reiss's enunciation is as follows (p. 209):—

"Théorème V.—Soient les échelles

$$\begin{pmatrix} a\ b \ldots r, & a\ , & b\ , & \ldots\ r \\ 1\ 2 \ldots n, & n+1, & n+2, & \ldots\ 2n \end{pmatrix} \text{ et } \begin{pmatrix} \alpha\ \beta\ \gamma \ldots \alpha^n, & \alpha^{n+1}, & \ldots\ \rho \\ 1\ 2\ 3 \ldots n\ , & n+1, & \ldots\ 2n \end{pmatrix},$$

qu'on fasse avec les élémens $\beta, \gamma, \ldots, \rho$ toutes les combinai-

* It is worthy of note in passing, that a common method does exist for establishing the two cases,—a method quite analogous to Reiss's, but difficult of suggestion to one who used his notation, or indeed to any one who had no notation suitable for determinants whose elements had special numerical values. All the change necessary is to make the last n elements of the first column each equal to zero. This causes no difference in the result when n is even, *e.g.*, from the identity

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \cdot & a_2 & a_3 & a_4 \\ \cdot & b_2 & b_3 & b_4 \end{vmatrix} = 0$$

we have, as before,

$$|a_1b_2|\cdot|a_3b_4| - |a_1b_3|\cdot|a_2b_4| + |a_1b_4|\cdot|a_2b_3| = 0;$$

and when n is odd, the second half of the terms which previously gave trouble do not occur.

sons de la classe $(n-1)$, et qu'on les substitue successivement dans le premier facteur du produit

$$(ab\ldots r\,,\,a\beta\gamma\ldots a^n)\,.\,(ab\ldots r\,,\;a^{n+1}\ldots\rho)$$

au lieu de $\beta\gamma\ldots a^n$; qu'on remplace maintenant dans l'autre facteur les exposans $a^{n+1}\ldots\rho$ par tous ceux qui ne se trouvent pas dans le premier, en ayant soin de les écrire suivant l'ordre indiqué par les échelles. Si l'on donne au premier produit le signe $(+)$, et qu'on détermine les signes de tous les autres d'après (II), la somme algébrique en sera $=0$, que le nombre n soit pair ou impair." (XXIII. 8)

An example of it is

$$\begin{aligned}
&(abc,\,123)(abc,\,456) - (abc,\,124)(abc,\,356)\\
&+(abc,\,125)(abc,\,346) - (abc,\,126)(abc,\,345)\\
&+(abc,\,134)(abc,\,256) - (abc,\,135)(abc,\,246)\\
&+(abc,\,136)(abc,\,245) + (abc,\,145)(abc,\,236)\\
&-(abc,\,146)(abc,\,235) + (abc,\,156)(abc,\,234) = 0,
\end{aligned}$$

the left-hand side being nothing more than the first ten terms of one of the expansions of the vanishing determinant

$$\begin{vmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6\\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6\\
c_1 & c_2 & c_3 & c_4 & c_5 & c_6\\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6\\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6\\
c_1 & c_2 & c_3 & c_4 & c_5 & c_6
\end{vmatrix},$$

or the other ten terms with their signs changed. Reiss's proof is lengthy and troublesome, the method being to expand each factor in terms of the a's and their complementary minors, perform the multiplications (*e.g.*, in the special case just given the multiplication of $a_1|b_2c_3| - a_2|b_1c_3| + a_3|b_1c_2|$ by $a_4|b_5c_6| - a_5|b_4c_6| + a_6|b_4c_5|$, &c.) and show that the terms of the final aggregate occur in pairs which annul themselves.

The next theorem is of still greater interest, because it is that

peculiar generalisation of the preceding which in later times came to be known as the *Extensional.* The way in which it is established is also noteworthy, viz., by deducing it as a special case from the theorem of which, as we have said, it may be viewed as a generalisation. The authors words are (p. 213) :—

"Ce théorème nous conduit à une relation qui existe dans le cas le plus général, savoir si $\nu - n$ est un nombre quelconque ou positif ou négatif. Supposons $\nu > n$, et $\nu - n = \mathrm{N}$; soient les échelles,

$$\begin{pmatrix} a\,b \ldots r\,, & a\,, & b\,, & \ldots r\,, & \mathrm{A}\,, & \mathrm{B}\,, & \ldots \mathrm{R} \\ 1\,2 \ldots \mathrm{N}\,, & \mathrm{N}+1\,, & \mathrm{N}+2\,, & \ldots 2\mathrm{N}\,, & 2\mathrm{N}+1\,, & 2\mathrm{N}+2\,, & \ldots \nu \end{pmatrix}$$

et

$$\begin{pmatrix} \alpha\,\beta \ldots \alpha^{\mathrm{N}}\,, & \alpha^{\mathrm{N}+1}\,, & \ldots \rho\,, & \mathrm{A}\,, & \mathrm{B}\,, & \ldots \mathrm{P} \\ 1\,2 \ldots \mathrm{N}\,, & \mathrm{N}+1\,, & \ldots 2\mathrm{N}\,, & 2\mathrm{N}+1\,, & 2\mathrm{N}+2\,, & \ldots \nu \end{pmatrix}.$$

Qu'on fasse avec les élémens $\beta, \gamma, \ldots \alpha^{\mathrm{N}}, \alpha^{\mathrm{N}+1} \ldots \rho$ toutes les combinaisons de la classe $\mathrm{N}-1$; qu'on les substitue successivement au lieu de $\beta \ldots \alpha^{\mathrm{N}}$ dans le premier facteur du produit

$$\begin{aligned} &(ab \ldots r\mathrm{AB} \ldots \mathrm{R},\ \alpha\beta \ldots \alpha^{\mathrm{N}}\mathrm{AB} \ldots \mathrm{P}) \\ \times &(ab \ldots r\mathrm{AB} \ldots \mathrm{R},\ \alpha^{\mathrm{N}+1} \ldots \rho\,\mathrm{AB} \ldots \mathrm{P})\,; \end{aligned}$$

qu'on remplace dans l'autre facteur les exposans $\alpha^{\mathrm{N}+1} \ldots \rho$ par tous ceux qui ne se trouvent pas dans le premier: qu'on détermine enfin le signe de chaque produit d'après (II): la somme algébrique en sera $= 0$. (XXIII. 9) (XLV. 6)

"En effet, supposons les échelles

$$\begin{pmatrix} a\,b \ldots r\,, & \mathrm{A}\,, & \mathrm{B}\,, & \ldots \mathrm{R}\,, & a\,, & b\,, & \ldots r\,, & \mathrm{A}\,, & \mathrm{B}\,, & \ldots \mathrm{R} \\ 1\,2 \ldots \mathrm{N}\,, & \mathrm{N}+1\,, & \mathrm{N}+2\,, & \ldots \nu-\mathrm{N}\,, & \nu-\mathrm{N}+1\,, & \nu-\mathrm{N}+2, & \ldots \nu\,, & \nu+1\,, & \nu+2\,, & \ldots 2\nu-2\mathrm{N} \end{pmatrix}$$

et

$$\begin{pmatrix} \alpha\beta \ldots \alpha^{\mathrm{N}}, & \alpha^{\mathrm{N}+1}\,, & \ldots \rho\,, & \mathrm{A}, & \mathrm{B}, & \ldots \mathrm{A}^{\nu-3\mathrm{N}}, & \mathrm{A}^{\nu-3\mathrm{N}+1}\,, & \ldots \mathrm{P}, & \mathrm{A}\,, & \ldots \mathrm{P} \\ 1\,2 \ldots \mathrm{N}\,, & \mathrm{N}+1\,, & \ldots 2\mathrm{N}, & 2\mathrm{N}+1\,, & 2\mathrm{N}+2\,, & \ldots \nu-\mathrm{N}\,, & \nu-\mathrm{N}+1\,, & \ldots \nu\,, & \nu+1\,, & \ldots 2\nu-2\mathrm{N} \end{pmatrix}.$$

Formons avec ces élémens la fonction décrite dans le dernier théorème : la somme totale en sera donc $= 0$, et le premier terme aura la forme

$$(ab\ldots r\mathrm{AB}\ldots\mathrm{R},\ \alpha\beta\ldots\rho\mathrm{A}\ldots\mathrm{A}^{\nu-3\mathrm{N}})$$
$$\times(ab\ldots r\mathrm{AB}\ldots\mathrm{R},\ \mathrm{A}^{\nu-3\mathrm{N}+1}\ldots\mathrm{PA}\ldots\mathrm{P}).$$

Or, on voit facilement que tous les termes qui ne contiennent pas dans chaque facteur *tous* les exposans A,B, . . . P, s'évanouiront séparément, parce qu'il y aura des exposans identiques dans l'un ou l'autre des facteurs. Il ne restera donc que les termes qui, contenant α dans le premier facteur, y épuisent successivement toutes les combinaisons de la classe N−1 des élémens $\beta, \gamma, \ldots \rho$. Mais les signes de ces termes sont évidemment déterminés comme ils devaient l'être; partant la somme algébrique de tous les termes est = 0, ce qu'il fallait démontrer.

This will be best understood by considering a special example. Going back to the previous theorem, and selecting its simplest case, we have

$$|a_1b_2|\cdot|a_3b_4| - |a_1b_3|\cdot|a_2b_4| + a_1b_4|\cdot|a_2b_3| = 0.$$

Now what the new theorem asserts in regard to this is that we may with impunity *extend* each of the determinants occurring in it, provided the extension be the same throughout. For example, choosing the extension $\xi_5\zeta_6\eta_7$,* we can, in virtue of the new theorem, assert the truth of the identity

$$|a_1b_2\xi_5\eta_6\zeta_7|\cdot|a_3b_4\xi_5\eta_6\zeta_7| - |a_1b_3\xi_5\eta_6\zeta_7|\cdot|a_2b_4\xi_5\eta_6\zeta_7|$$
$$+ |a_1b_4\xi_5\eta_6\zeta_7|\cdot|a_2b_3\xi_5\eta_6\zeta_7| = 0.$$

That the two may be viewed as cases of the same theorem will be apparent when it is pointed out that just as the first is derivable from

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ . & a_2 & a_3 & a_4 \\ . & b_2 & b_3 & b_4 \end{vmatrix} = 0,$$

* In Reiss's notation the extension is $\mathrm{A_A\,B_B}\ldots\mathrm{R_P}$.

so the second is derivable in exactly the same way from a perfectly similar identity,* viz.

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_5 & b_6 & b_7 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_5 & \xi_6 & \xi_7 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_5 & \eta_6 & \eta_7 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \zeta_7 & \zeta_5 & \zeta_6 & \zeta_7 \\ \cdot & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_5 & a_6 & a_7 \\ \cdot & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_5 & b_6 & b_7 \\ \cdot & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \xi_5 & \xi_6 & \xi_7 \\ \cdot & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_5 & \eta_6 & \eta_7 \\ \cdot & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \xi_7 & \zeta_5 & \zeta_6 & \zeta_7 \end{vmatrix} = 0.$$

Many more products than three (126 in fact) arise in the latter case; but, for the reason stated by Reiss, only three of them do not vanish.

JACOBI (1829, 1830).

[Exercitatio algebraica circa discerptionem singularem fractionum, quae plures variabiles involvunt. *Crelle's Journal*, v. pp. 344–364.]

[De resolutione aequationum per series infinitas. *Crelle's Journal*, vi. pp. 257–286.]

By such memoirs as these, in which Jacobi continued to use determinants, the functions were kept before the mathematical

* It is perhaps a little more readily seen to be derivable from

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & \cdot & \cdot & \cdot \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & \cdot & \cdot & \cdot \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 & \cdot & \cdot & \cdot \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \cdot & \cdot & \cdot \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \zeta_7 & \cdot & \cdot & \cdot \\ \cdot & a_2 & a_3 & a_4 & \cdot & \cdot & \cdot & a_5 & a_6 & a_7 \\ \cdot & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot & b_5 & b_6 & b_7 \\ \cdot & \xi_2 & \xi_3 & \xi_4 & \cdot & \cdot & \cdot & \xi_5 & \xi_6 & \xi_7 \\ \cdot & \eta_2 & \eta_3 & \eta_4 & \cdot & \cdot & \cdot & \eta_5 & \eta_6 & \eta_7 \\ \cdot & \zeta_2 & \zeta_3 & \zeta_4 & \cdot & \cdot & \cdot & \zeta_5 & \zeta_6 & \zeta_7 \end{vmatrix} = 0.$$

world. For the present it will suffice to note in regard to them that although general determinants in Laplace's notation occur (p. 351, &c.), the real interest of the papers arises from the fact that use is made in them of that special form which afterwards came to be associated with Jacobi's name. His introductory words concerning it are as follows (pp. 348, 349):—

"Vocemus porro Δ determinantem differentialium partialium sequentium :

$$\begin{array}{cccccc}
\frac{\partial u}{\partial x}, & \frac{\partial u}{\partial x_1}, & \frac{\partial u}{\partial x_2}, & \cdot\ \cdot\ \cdot\ \cdot & , & \frac{\partial u}{\partial x_{n-1}} \\
\frac{\partial u_1}{\partial x}, & \frac{\partial u_1}{\partial x_1}, & \frac{\partial u_1}{\partial x_2}, & \cdot\ \cdot\ \cdot\ \cdot & , & \frac{\partial u_1}{\partial x_{n-1}} \\
\cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot & \cdot \\
\frac{\partial u_{n-1}}{\partial x}, & \frac{\partial u_{n-1}}{\partial x_1}, & \frac{\partial u_{n-1}}{\partial x_2}, & \cdot\ \cdot\ \cdot\ \cdot & , & \frac{\partial u_{n-1}}{\partial x_{n-1}}.
\end{array}$$

Erit *e.g.* pro tribus functionibus u, u_1, u_2, tribusque variabilibus x, y, z :

$$\Delta = \frac{\partial u}{\partial x}\cdot\frac{\partial u_1}{\partial y}\cdot\frac{\partial u_2}{\partial z} - \frac{\partial u}{\partial x}\cdot\frac{\partial u_1}{\partial z}\cdot\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\cdot\frac{\partial u_2}{\partial x}\cdot\frac{\partial u}{\partial z}$$
$$-\frac{\partial u_2}{\partial z}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial u_1}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial u_1}{\partial z}\cdot\frac{\partial u_2}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial u_1}{\partial x}\cdot\frac{\partial u_2}{\partial y},$$

quam patet expressionem casu, quo u, u_1, u_2 sunt expressiones lineares, in expressionem ipsius Δ supra exhibitam redire."

MINDING (1829).

[Auflösung einiger Aufgaben der analytischen Geometrie vermittelst des barycentrischen Calculs. *Crelle's Journal*, v. pp. 397–401.]

Unlike Jacobi, Minding was unaware, apparently, of the existence of a theory of determinants. The functions occur at every step of his investigation, yet he makes no use of their known properties to obtain his results.

He deals with four problems in his memoir, the second two being the analogues, in space, of the first two. Nothing noteworthy

occurs in connection with the latter save that use is made of the identity,

$$\frac{\beta'\gamma''-\beta''\gamma'}{a'} = a(b'c''-b''c') + a'(b''c-bc'') + a''(bc'-b'c),$$

where
$$\beta' = ba'-b'a, \qquad \beta'' = b'a''-b''a',$$
$$\gamma' = ca'-c'a, \qquad \gamma'' = c'a''-c''a'.$$

This identity, it may be remembered, we have noted under Lagrange as an elementary case of the theorem afterwards well known regarding a minor of the adjugate determinant. Strange to say, it makes only its second appearance here fifty-six years afterwards. In the interim, too, no other special case of the theorem seems to have been established.

The third is that if P, P′, P″, P‴, be four points in space, given by the equations,

$$\begin{aligned} q\,\mathrm{P} &= a\,\mathrm{A} + b\,\mathrm{B} + c\,\mathrm{C} + d\,\mathrm{D}, \\ q'\,\mathrm{P}' &= a'\,\mathrm{A} + b'\,\mathrm{B} + c'\,\mathrm{C} + d'\,\mathrm{D}, \\ q''\,\mathrm{P}'' &= a''\,\mathrm{A} + b''\,\mathrm{B} + c''\,\mathrm{C} + d''\,\mathrm{D}, \\ q'''\,\mathrm{P}''' &= a'''\,\mathrm{A} + b'''\,\mathrm{B} + c'''\,\mathrm{C} + d'''\,\mathrm{D}; \end{aligned}$$

then for the bulk of the tetrahedron P P′ P″ P‴, we have

$$\frac{\mathrm{PP'P''P'''}}{\mathrm{ABC\,D}} = \frac{\mathrm{A}+\mathrm{A}'+\mathrm{A}''}{qq'q''q'''},$$

where

$$\mathrm{A} = \delta'(\beta''\gamma'''-\beta'''\gamma''), \quad \mathrm{A}' = \delta''(\beta'''\gamma'-\beta'\gamma'''), \quad \mathrm{A}'' = \delta'''(\beta'\gamma''-\beta''\gamma'),$$

and

$$\begin{aligned} \beta' &= a'\,b - a\,b', & \gamma' &= a'\,c - a\,c', & \delta' &= a'\,d - a\,d', \\ \beta'' &= a''\,b' - a'\,b'', & \gamma'' &= a''\,c' - a'\,c'', & \delta'' &= a''\,d' - a'\,d'', \\ \beta''' &= a'''b'' - a''b''', & \gamma''' &= a'''c'' - a''c''', & \delta''' &= a'''d'' - a''d'''. \end{aligned}$$

The transformation of $\mathrm{A}+\mathrm{A}'+\mathrm{A}''$ into the form

$$a'a''|ab'c''d'''|$$

—a transformation all-important for Minding's purpose — is not made: but in the remark,

"Man kann den Ausdruck $A + A' + A''$ leicht entwickeln, und wird ihn dann durch $a'a''$ theilbar finden,"

there is evidently a foreshadowing of the identity

$$\begin{vmatrix} |a'\,b\,|, & |a'\,c\,|, & |a'\,d\,| \\ |a''\,b'|, & |a''\,c'|, & |a''\,d'| \\ |a'''b''|, & |a'''c''|, & |a'''d''| \end{vmatrix} = -a'a''|ab'c''d'''| .$$

The fourth theorem, concerning the tetrahedron enclosed by four given planes,

$$\begin{array}{l} A + xB + yC + (a \;\; + b \;\; x + c \;\; y)C , \\ A + xB + yC + (a' \;\; + b' \;\; x + c' \;\; y)C , \\ A + xB + yC + (a'' + b'' x + c'' y)C , \\ A + xB + yC + (a''' + b'''x + c'''y)C , \end{array}$$

is made dependent on the third. The intersections Π, Π', Π'', Π''' of the four triads of planes are found to be given by

$$\begin{array}{lllll} q\;\Pi &= (b\;c'\,)A &+ (c\;a'\,)B &+ (a\;b'\,)C &+ (a\;b\;c\;)D , \\ q'\;\Pi' &= (b'\;c''\,)A &+ (c'\;a''\,)B &+ (a'\;b''\,)C &+ (a'\;b'\;c'\,)D , \\ q''\,\Pi'' &= (b''\,c''')A &+ (c''\,a''')B &+ (a''\,b''')C &+ (a''\,b''\,c''\,)D , \\ q'''\Pi''' &= (b'''c\;)A &+ (c'''a\;)B &+ (a'''b\;)C &+ (a'''b'''c''')D , \end{array}$$

where

$$\begin{aligned} (bc') &= b(c' - c'') + b'(c'' - c) + b''(c - c') , \\ (ca') &= c(a' - a'') + c'(a'' - a) + c''(a - a') , \\ (ab') &= a(b' - b'') + a'(b'' - b) + a''(b - b') , \end{aligned}$$

and

$$\begin{aligned} (abc) &= a(bc') + b(ca') + c(ab') , \\ &= a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c) . \end{aligned}$$

Hence, by the third theorem,

$$\frac{\Pi\Pi'\Pi''\Pi'''}{A B C\;D} = \frac{A + A' + A''}{qq'q''q'''(b'c'')(b''c''')} ,$$

where now

$$A = \delta'(\beta''\gamma''' - \beta'''\gamma''), \quad A' = \delta''(\beta'''\gamma' - \beta'\gamma'''), \quad A'' = \delta'''(\beta'\gamma'' - \beta''\gamma') ,$$

and

$$\beta' = (b'c'')(ca') - (bc')(c'a''), \quad \beta'' = \ldots, \quad \beta''' = \ldots,$$
$$\gamma' = (b'c'')(ab') - (a'b'')(bc'), \quad \gamma'' = \ldots, \quad \gamma''' = \ldots,$$
$$\delta' = (b'c'')(abc) - (bc')(a'b'c'), \quad \delta'' = \ldots, \quad \delta''' = \ldots.$$

Minding then continues (pp. 399, 400) :—

"Man setze

$$a'''(bc') - a(b'c'') + a'(b''c''') - a''(b'''c) = \mathrm{M}.$$

"Nach den nöthigen Reductionen erhält man:

$$\beta' = -(c''-c')\mathrm{M}, \quad \gamma' = -(b'-b'')\mathrm{M}, \quad \delta' = -(b'c''-b''c')\mathrm{M},$$
$$\beta'' = +(c'''-c'')\mathrm{M}, \quad \gamma'' = +(b''-b''')\mathrm{M}, \quad \delta'' = +(b''c'''-b'''c'')\mathrm{M},$$
$$\beta''' = -(c-c''')\mathrm{M}, \quad \gamma''' = -(b'''-b)\mathrm{M}, \quad \delta''' = -(b'''c-bc''')\mathrm{M}.$$

"Hieraus erhält man weiter:

$$\begin{aligned} \mathrm{A} &= -\mathrm{M}^3(b''c'-b'c'').(b''c'''), \\ \mathrm{A}' &= -\mathrm{M}^3(b'''c''-b''c''').\{(b''c''')-(b'''c)\}, \\ \mathrm{A}'' &= -\mathrm{M}^3(bc'''-b'''c).(b'c''). \end{aligned}$$

"Eine weitere Reduction ergiebt:

$$(bc'''-b'''c)(b'c'') - (b'''c)(b'''c''-b''c''') = (c'''b'-c'b''').$$

"Hieraus folgt $\mathrm{A}+\mathrm{A}'+\mathrm{A}'' = \mathrm{M}^3(b'c'')(b''c''')$, und als Resultat:

$$\frac{\Pi\Pi'\Pi''\Pi'''}{\mathrm{A\,B\,C\,D}} = \frac{\mathrm{M}^3}{qq'q''q'''}."$$

The first point to be noted here is, that since

$$(bc'), \quad (ca'), \quad (ab'),$$

are in modern notation

$$\begin{vmatrix} b & b' & b'' \\ c & c' & c'' \\ 1 & 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} c & c' & c'' \\ a & a' & a'' \\ 1 & 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ 1 & 1 & 1 \end{vmatrix},$$

the identity

$$a(bc') + b(ca') + c(ab') = a(b'c''-b''c') + a'(b''c-bc'') + a''(bc'-b'c)$$

is the same as

$$a\begin{vmatrix} b & b' & b'' \\ c & c' & c'' \\ 1 & 1 & 1 \end{vmatrix} + b\begin{vmatrix} c & c' & c'' \\ a & a' & a'' \\ 1 & 1 & 1 \end{vmatrix} + c\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix},$$

—a disguised special case of Vandermonde's theorem (XII.), the four elements of one row being each unity. (XII. 11)

The next point is, that since the expression denoted by M, viz.,

$$a'''(bc') - a(b'c'') + a'(b''c''') - a''(b'''c)$$

is in modern notation

$$-\begin{vmatrix} a & a' & a'' & a''' \\ b & b' & b'' & b''' \\ c & c' & c'' & c''' \\ 1 & 1 & 1 & 1 \end{vmatrix},$$

the identity

$$\delta' = -(b'c'' - b''c')\mathrm{M}$$

is the same as

$$\begin{Vmatrix} \begin{vmatrix} b' & b'' & b''' \\ c' & c'' & c''' \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} b & b' & b'' \\ c & c' & c'' \\ 1 & 1 & 1 \end{vmatrix} \\ \begin{vmatrix} a' & a'' & a''' \\ b' & b'' & b''' \\ c' & c'' & c''' \end{vmatrix} & \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} \end{Vmatrix} = \begin{vmatrix} b' & b'' \\ c' & c'' \end{vmatrix} \cdot \begin{vmatrix} a & a' & a'' & a''' \\ b & b' & b'' & b''' \\ c & c' & c'' & c''' \\ 1 & 1 & 1 & 1 \end{vmatrix},$$

and therefore is, like its eight companions, a fresh case of the theorem regarding a minor of the adjugate.* (XX. 2)

DRINKWATER, J. E. (1831).

[On Simple Elimination. *Philosophical Magazine*, x. pp. 24–28.]

Up to this date, almost 140 years after the publication of Leibnitz's letter to De L'Hôpital, no English mathematician's name

* Instead of following Minding's lengthy process, a mathematician of the present time would of course observe that the coefficients of A, B, C, D are the principal minors of M, and using Cauchy's theorem would at once reach the desired conclusion, viz., that the determinant of them $= \mathrm{M}^3$.

occurs in connection with the subject of determinants,—a fact most significant of the comparative neglect of mathematical studies in Britain during the 18th century. Apart from the contents, therefore, some little interest attaches to Drinkwater's short paper, as being the first sign to us of that revival which, as is well known otherwise, had taken place some few years before.

Drinkwater knew of the investigations of Cramer, Bézout, and Laplace; and professed only to put the elements of the subject "in a more convenient form." His rule of signs is stated and illustrated as follows (p. 25):—

> "Write down the series of natural numbers 1 2 3 4 . . . n, and underneath it all the permutations of these n numbers, prefixing to each a positive or negative sign according to the following condition:—
>
> "Any permutation may be derived from the first by considering a requisite number of figures to move from left to right by a certain number of single steps or descents of a single place. If the whole number of such single steps necessary to derive any permutation from the first be even, that permutation has a positive sign prefixed to it; the others are negative. For instance, 4 2 1 3 . . . n may be derived from 1 2 3 4 . . . . n, by first causing the 3 to descend below the 4, requiring one single step: then the 2 below the new place of the 4, another single step; lastly, the 1 below the new place of the 2, requiring two more steps, making in all 4. Therefore this permutation requires the positive sign."

In this there is essentially nothing new: it at once recalls a theorem of Rothe's (III. 8). In the following paragraph, however, we find the discussion of a point not previously dealt with. The words are (p. 25):—

> "The same permutation may be derived in various ways, and it is necessary, therefore, to show that this rule is not inconsistent with itself: thus the same permutation 4 2 1 3. . . n might have been obtained by first marching 1 through three places, then 2 through two; and, lastly, 3 through one, making six in all, an even number as before. Without accumulating instances, it is plain, if q be the smallest number of

steps by which any number p reaches the place it is intended finally to occupy in that permutation, that if p should advance in the first instance m places beyond this, it must subsequently return through m places: or, which is the same thing, it must at a later period of the march, allow m of those which it has passed to repass it, so that it will regain its proper place after the number of steps has been increased from q to $q + 2m$, which, by the rule, require the same sign as q. The same reasoning applies to every other figure; and hence the consistency of the rule is evident." (III. 25)

He then establishes four properties of the functions, viz. (1) Vandermonde's theorem regarding the effect produced on the *function* by transposition of a pair of letters; (2) Bézout's recurrent law of formation; (3) Scherk's theorem regarding the partition of one of the functions into two; and (4) Scherk's theorem regarding the removal of a constant factor from one of the functions. The two latter theorems, which, as we have seen, had been stated for the first time only six years before, are given by Drinkwater in the following form (p. 27):—

"(8) If any factor in $f\{XYZT\ .\ .\ .\ (n)\}$, as X, be divided into two parts, $X = V + W$, the function may be similarly divided, so that

$$f\{(V+W)YZT\ .\ .\ .\ (n)\} = f\{VYZT\ .\ .\ .\ (n)\} + f(WYZT\ .\ .\ .\ (n)\},$$

placing each part of X in the same relative position (which in this example is the first) which X itself occupied before the division. (XLVII. 2)

(9) If any quantity which does not vary from one equation to the other, and which, therefore, is not liable to be affected with an index, is found under the symbol, it may be considered a constant coefficient of every term of the developed function; and written as such on the outside of the symbol: of this nature are the unknown quantities themselves, so that for instance,

$$f\{XYxZT\ .\ .\ .\ .\ (n)\} = xf\{XYZT\ .\ .\ .\ (n)\},$$

and so of like quantities." (XLVIII. 2)

After these preliminaries the problem of the solution of n linear

equations in n unknowns is taken up. The method followed is essentially the same as Scherk's.

MAINARDI (1832).

[Trasformazioni di alcune funzioni algebraiche, e loro uso nella geometria e nella meccanica. Memoria di Gaspare Mainardi. 44 pp. Pavia, 1832.]

In his preface Mainardi explains that the algebraical functions referred to in the title are "*funzioni risultanti* o *determinanti.*" But although he thus speaks of them as if they were known to mathematicians by name, and mentions the researches of Monge, Lagrange, Cauchy, and Binet in regard to them, he does not take for granted that his reader has a knowledge of any of their properties. The one theorem on determinants,—the multiplication-theorem,—which forms the basis of the whole memoir, is consequently sought to be established without the use of any previously proved theorem. The attempt, as might be expected, is interesting.

The first two sections (pp. 9–29) of the three into which the memoir is divided may be passed over without much comment. The first deals with the multiplication-theorem for two determinants of the 2nd order, and with those applications of it to geometry which arise on making the elements of each determinant the Cartesian co-ordinates of two points in a plane. No proof is considered necessary for this simple case, the opening paragraph of the memoir being;—

> "Rappresentate con x_m, x_n, x_a, x_b; y_m, y_n, y_a, y_b otto quantità qualsivogliano, ed indicati per brevità il binomio
>
> $$x_m \,.\, x_a + y_m \,.\, y_a \quad \text{col simbolo} \quad (x_m x_a)\,,$$
>
> il binomio
>
> $$x_n \,.\, x_b + y_n \,.\, y_b \qquad \text{con} \qquad (x_n x_b)$$
>
> e simili, si proverà facilmente essere
>
> $$\text{(a)} \qquad \begin{aligned} &(x_m y_n - x_n y_m)(x_a y_b - x_b y_a) \\ &= (x_m x_a)(x_n x_b) - (x_m x_b)(x_n x_a)." \end{aligned}$$

All the seven other paragraphs are geometrical.

The second section in like manner opens with an algebraical theorem, viz. (p. 13)—

$$
\begin{aligned}
&\{x_m(y_p - y_n)\}\{x_a(y_c - y_b)\} \\
&+ \{x_m(z_p - z_n)\}\{x_a(z_c - z_b)\} \\
&+ \{y_m(z_p - z_n)\}\{y_a(z_c - z_b)\} \\
= \; & (x_m x_a)(x_p x_c) - (x_m x_c)(x_p x_a) + (x_n x_a)(x_m x_c) \\
& - (x_n x_c)(x_m x_a) + (x_p x_a)(x_n x_c) - (x_p x_c)(x_n x_a) \\
& + (x_m x_b)(x_p x_a) - (x_m x_a)(x_p x_b) + (x_n x_b)(x_m x_a) \\
& - (x_n x_a)(x_m x_b) + (x_p x_b)(x_n x_a) - (x_p x_a)(x_n x_b) \\
& + (x_m x_c)(x_p x_b) - (x_m x_b)(x_p x_c) + (x_n x_c)(x_m x_b) \\
& - (x_n x_b)(x_m x_c) + (x_p x_c)(x_n x_b) - (x_p x_b)(x_n x_c),
\end{aligned}
\tag{XXIX. 2}
$$

where $\{x_m(y_p - y_n)\}$ and $(x_m x_a)$ stand for

$$(x_m y_p - x_p y_m) + (x_n y_m - x_m y_n) + (x_p y_n - x_n y_p)$$

and

$$x_m x_a + y_m y_a + z_m z_a$$

respectively; and the remainder is occupied with the applications of the theorem to geometry and dynamics. Each factor of the left-hand side of the identity is evidently a determinant of the third order, and the three pairs of lines on the right-hand side are each the expansion of a determinant of the same order: so that in the notation of the present day the identity may be written

$$
\begin{aligned}
&\begin{vmatrix} x_m & y_m & 1 \\ x_n & y_n & 1 \\ x_p & y_p & 1 \end{vmatrix} . \begin{vmatrix} x_a & y_a & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{vmatrix} + \begin{vmatrix} x_m & z_m & 1 \\ x_n & z_n & 1 \\ x_p & z_p & 1 \end{vmatrix} . \begin{vmatrix} x_a & z_a & 1 \\ x_b & z_b & 1 \\ x_c & z_c & 1 \end{vmatrix} \\
&+ \begin{vmatrix} y_m & z_m & 1 \\ y_n & z_n & 1 \\ y_p & z_p & 1 \end{vmatrix} . \begin{vmatrix} y_a & z_a & 1 \\ y_b & z_b & 1 \\ y_c & z_c & 1 \end{vmatrix} = \begin{vmatrix} (x_m x_c) & (x_m x_a) & 1 \\ (x_n x_c) & (x_n x_a) & 1 \\ (x_p x_c) & (x_p x_a) & 1 \end{vmatrix} \\
&\qquad + \begin{vmatrix} (x_m x_a) & (x_m x_b) & 1 \\ (x_n x_a) & (x_n x_b) & 1 \\ (x_p x_a) & (x_p x_b) & 1 \end{vmatrix} \\
&\qquad + \begin{vmatrix} (x_m x_b) & (x_m x_c) & 1 \\ (x_n x_b) & (x_n x_c) & 1 \\ (x_p x_b) & (x_p x_c) & 1 \end{vmatrix} .
\end{aligned}
$$

There has been no previous instance of an identity perfectly similar to this; the nearest approach to such being, as the numbering shows, a result obtained by Binet in 1811. The exact character of the affinity between the two, and the general theorem which both foreshadow, will be most readily brought into evidence by a little additional transformation. Taking first the right-hand side of the identity, we observe that the three determinants have only twelve elements among them, being obtainable in fact from a single array of four rows and three columns. Their sum may consequently be put in the form

$$\begin{vmatrix} 1 & (x_m x_a) & (x_m x_b) & (x_m x_c) \\ 1 & (x_n x_a) & (x_n x_b) & (x_n x_c) \\ 1 & (x_p x_a) & (x_p x_b) & (x_p x_c) \\ 0 & 1 & 1 & 1 \end{vmatrix}.$$

Secondly, we observe that the first factors on the left-hand side are similarly obtainable from

$$\begin{matrix} x_m & y_m & z_m & 1 \\ x_n & y_n & z_n & 1 \\ x_p & y_p & z_p & 1\,; \end{matrix}$$

and the second factors from

$$\begin{matrix} x_a & y_a & z_a & 1 \\ x_b & y_b & z_b & 1 \\ x_c & y_c & z_c & 1\,; \end{matrix}$$

and as the so-called product of these arrays is equal to the said left-hand member diminished by

$$\begin{vmatrix} x_m & y_m & z_m \\ x_n & y_n & z_n \\ x_p & y_p & z_p \end{vmatrix} \cdot \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix},$$

Mainardi's theorem may be put in the much altered form—

$$\begin{vmatrix} 1 & (x_m x_a) & (x_n x_a) & (x_p x_a) \\ 1 & (x_m x_b) & (x_n x_b) & (x_p x_b) \\ 1 & (x_m x_c) & (x_n x_c) & (x_p x_c) \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x_m & y_m & z_m & 1 \\ x_n & y_n & z_n & 1 \\ x_p & y_p & z_p & 1 \end{vmatrix} \cdot \begin{vmatrix} x_a & y_a & z_a & 1 \\ x_b & y_b & z_b & 1 \\ x_c & y_c & z_c & 1 \end{vmatrix}$$

$$- \begin{vmatrix} x_m & y_m & z_m \\ x_n & y_n & z_n \\ x_p & y_p & z_p \end{vmatrix} \cdot \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}.$$

The constitution of the 3rd section is quite like that of the others, the first paragraph dealing with the multiplication-theorem for the case of determinants of the 3rd order, the second paragraph with the same theorem for determinants of the 4th order, and the remaining eight paragraphs with geometrical applications. The mode of proof of the multiplication-theorem is partly indicated by saying that any particular case is made dependent on the case immediately preceding it; but its exact character can only be understood by a somewhat minute examination. The investigation for the case of determinants of the 3rd order stands as follows (p. 29):—

"Si considerino i due polinomj

$$(l)\quad \begin{aligned} x_m(y_nz_p - y_pz_n) + x_n(z_my_p - y_mz_p) + x_p(y_mz_n - y_nz_m) \\ = \{x_m, y_n, z_p\}, \\ x_a(y_bz_c - y_cz_b) + x_b(z_ay_c - z_cy_a) + x_c(y_az_b - y_bz_a) \\ = \{x_a, y_b, z_c\}. \end{aligned}$$

Se ne effettui il prodotto, il quale, mediante l'equazione (a) del primo articolo, si potrà disporre sotto la forma seguente

$$(h)\quad \begin{aligned} & x_mx_a(y_ny_b)(y_py_c) - x_mx_a(y_ny_c)(y_py_b) \\ + & x_nx_a(y_my_c)(y_py_b) - x_nx_a(y_my_b)(y_py_c) \\ + & x_px_a(y_my_b)(y_ny_c) - x_px_a(y_my_c)(y_ny_b) \\ + & x_mx_b(y_ny_c)(y_py_a) - x_mx_b(y_ny_a)(y_py_c) \\ + & x_nx_b(y_my_a)(y_py_c) - x_nx_b(y_my_c)(y_py_a) \\ + & x_px_b(y_my_c)(y_ny_a) - x_px_b(y_my_a)(y_ny_c) \\ + & x_mx_c(y_ny_a)(y_py_b) - x_mx_c(y_ny_b)(y_py_a) \\ + & x_nx_c(y_my_b)(y_py_a) - x_nx_c(y_my_a)(y_py_b) \\ + & x_px_c(y_my_a)(y_ny_b) - x_px_c(y_my_b)(y_ny_a). \end{aligned}$$

Esaminando ora la quantità

$$\begin{aligned} & x_mx_a\{x_nx_b(y_py_c) + x_px_c(y_ny_b) + x_nx_bx_px_c \\ & \qquad - x_nx_c(y_py_b) - x_px_b(y_ny_c) - x_nx_px_bx_c\} \\ + & x_nx_a\{x_mx_c(y_py_b) + x_px_b(y_my_c) + x_mx_cx_px_b \\ & \qquad - x_mx_b(y_py_c) - x_px_c(y_my_b) - x_mx_bx_px_c\} \\ + & x_px_a\{x_mx_b(y_ny_c) + x_nx_c(y_my_b) + x_mx_bx_nx_c \\ & \qquad - x_mx_c(y_ny_b) - x_nx_b(y_my_c) - x_mx_cx_nx_b\}, \end{aligned}$$

e le due espressioni che si traggono da questa, cambiando, prima a in b, b in c, c in a; poscia a in c, c in b, b in a; con facilità si scorge che la somma di questi polinomj è nulla identicamente, per cui si potrà aggiungere al prodotto (h) senza punto alterarlo. Fatta quest' addizione, l'aggregato altro non sarà che lo stesso polinomio (h), ove si supponga che i simboli $(y_n y_b)$, $(y_p y_c)$, ecc. rappresentino rispettivamente i trinomj seguenti

$$x_n x_b + y_n y_b + z_n z_b\,, \quad x_p x_c + y_p y_c + z_p z_c\,, \quad \text{ecc.}$$

Se ora si ordineranno le espressioni (l) portando fuori dalle parentesi y ovvero z in luogo di x, formeremo il prodotto delle medesime così scritte, ed opereremo come sopra, il risultato sarà il polinomio che si desume da (h) cambiando le x che sono fuori dalle parentesi in y ovvero in z egualmente accentate. Se faremo per ultimo la somma di queste tre espressioni, tal somma si caverà dal polinomio (h) scrivendo $(x_m x_a)$ ovvero $(y_m y_a)$ invece di $x_m x_a$; $(x_p x_a)$ in luogo di $x_p x_a$ ec. ec. e sarà eguale al triplo prodotto delle expressioni (l).

Essendo poi quella somma divisibile per tre, effettuata la divisione per questo numero, avremo

$$(1)\quad \{x_m, y_n, z_p\}.\{x_a, y_b, z_c\} = (x_m x_a)(x_n x_b)(x_p x_c) + (x_n x_a)(x_p x_b)(x_m x_c) + (x_p x_a)(x_m x_b)(x_n x_c) - (x_m x_a)(x_p x_b)(x_n x_c) - (x_n x_a)(x_m x_b)(x_p x_c) - (x_p x_a)(x_n x_b)(x_m x_c)."$$

(XVII. 6)

That the essential points of this method of demonstration may be seen, let us apply it as it would be applied if adopted at the present day.

The given determinants being

$$|a_1 b_2 c_3| \quad \text{and} \quad |\alpha_1 \beta_2 \gamma_3|,$$

we should say

$$|a_1 b_2 c_3| = a_1 |b_2 c_3| - a_2 |b_1 c_3| + a_3 |b_1 c_2|,$$

and
$$|\alpha_1 \beta_2 \gamma_3| = \alpha_1 |\beta_2 \gamma_3| - \alpha_2 |\beta_1 \gamma_3| + \alpha_3 |\beta_1 \gamma_2|;$$

hence, using the multiplication-theorem as established for determin-

ants of the 2nd order, and (to save on the breadth of the page) denoting

$$a\alpha + b\beta + c\gamma + \dots \quad \text{by} \quad \frac{a, b, c, \dots}{\alpha, \beta, \gamma, \dots}$$

we should have

$$|\, a_1 b_2 c_3 \,| \cdot |\, \alpha_1 \beta_2 \gamma_3 \,|$$

$$= a_1\alpha_1 \begin{vmatrix} \frac{b_2, c_2}{\beta_2, \gamma_2} & \frac{b_2, c_2}{\beta_3, \gamma_3} \\ \frac{b_3, c_3}{\beta_2, \gamma_2} & \frac{b_3, c_3}{\beta_3, \gamma_3} \end{vmatrix} - a_2\alpha_1 \begin{vmatrix} \frac{b_1, c_1}{\beta_2, \gamma_2} & \frac{b_1, c_1}{\beta_3, \gamma_3} \\ \frac{b_3, c_3}{\beta_2, \gamma_2} & \frac{b_3, c_3}{\beta_3, \gamma_3} \end{vmatrix} + a_3\alpha_1 \begin{vmatrix} \frac{b_1, c_1}{\beta_2, \gamma_2} & \frac{b_1, c_1}{\beta_3, \gamma_3} \\ \frac{b_2, c_2}{\beta_2, \gamma_2} & \frac{b_2, c_2}{\beta_3, \gamma_3} \end{vmatrix}$$

$$- a_1\alpha_2 \begin{vmatrix} \frac{b_2, c_2}{\beta_1, \gamma_1} & \frac{b_2, c_2}{\beta_3, \gamma_3} \\ \frac{b_3, c_3}{\beta_1, \gamma_1} & \frac{b_3, c_3}{\beta_3, \gamma_3} \end{vmatrix} + a_2\alpha_2 \begin{vmatrix} \frac{b_1, c_1}{\beta_1, \gamma_1} & \frac{b_1, c_1}{\beta_3, \gamma_3} \\ \frac{b_3, c_3}{\beta_1, \gamma_1} & \frac{b_3, c_3}{\beta_3, \gamma_3} \end{vmatrix} - a_3\alpha_2 \begin{vmatrix} \frac{b_1, c_1}{\beta_1, \gamma_1} & \frac{b_1, c_1}{\beta_3, \gamma_3} \\ \frac{b_2, c_2}{\beta_1, \gamma_1} & \frac{b_2, c_2}{\beta_3, \gamma_3} \end{vmatrix}$$

$$+ a_1\alpha_3 \begin{vmatrix} \frac{b}{\beta_1, \gamma_1} & \frac{b_2, c_2}{\beta_2, \gamma_2} \\ \frac{b_3, c_3}{\beta_1, \gamma_1} & \frac{b_3, c_3}{\beta_2, \gamma_2} \end{vmatrix} - a_2\alpha_3 \begin{vmatrix} \frac{b_1, c_1}{\beta_1, \gamma_1} & \frac{b_1, c_1}{\beta_2, \gamma_2} \\ \frac{b_3, c_3}{\beta_1, \gamma_1} & \frac{b_3, c_3}{\beta_2, \gamma_2} \end{vmatrix} + a_3\alpha_3 \begin{vmatrix} \frac{b_1, c_1}{\beta_1, \gamma_1} & \frac{b_1, c_1}{\beta_2, \gamma_2} \\ \frac{b_2, c_2}{\beta_1, \gamma_1} & \frac{b_2, c_2}{\beta_2, \gamma_2} \end{vmatrix}$$

That each line of this result is not altered in substance by writing

$$\frac{a_2, b_2, c_2}{\alpha_2, \beta_2, \gamma_2} \ \text{for}\ \frac{b_2, c_2}{\beta_2, \gamma_2}, \quad \frac{a_2, b_2, c_2}{\alpha_3, \beta_3, \gamma_3} \ \text{for}\ \frac{b_2, c_2}{\beta_3, \gamma_3}, \ \&\text{c.},$$

would probably be shown by expressing the line in the form of a determinant of the 3rd order, *e.g.*, the first line in the form

$$\alpha_1 \begin{vmatrix} a_1 & \frac{b_1, c_1}{\beta_2, \gamma_2} & \frac{b_1, c_1}{\beta_3, \gamma_3} \\ a_2 & \frac{b_2, c_2}{\beta_2, \gamma_2} & \frac{b_2, c_2}{\beta_3, \gamma_3} \\ a_3 & \frac{b_3, c_3}{\beta_2, \gamma_2} & \frac{b_3, c_3}{\beta_3, \gamma_3} \end{vmatrix};$$

and increasing each element of the second column by α_2 times the corresponding element of the first, and each element of the third column by α_3 times the corresponding element of the first. The whole result would in this way be transformed into

$$
\begin{vmatrix}
a_1\alpha_1 & \dfrac{a_1, b_1, c_1}{\alpha_2, \beta_2, \gamma_2} & \dfrac{a_1, b_1, c_1}{\alpha_3, \beta_3, \gamma_3} \\[2ex]
a_2\alpha_1 & \dfrac{a_2, b_2, c_2}{\alpha_2, \beta_2, \gamma_2} & \dfrac{a_2, b_2, c_2}{\alpha_3, \beta_3, \gamma_3} \\[2ex]
a_3\alpha_1 & \dfrac{a_3, b_3, c_3}{\alpha_2, \beta_2, \gamma_2} & \dfrac{a_3, b_3, c_3}{\alpha_3, \beta_3, \gamma_3}
\end{vmatrix}
-
\begin{vmatrix}
a_1\alpha_2 & \dfrac{a_1, b_1, c_1}{\alpha_1, \beta_1, \gamma_1} & \dfrac{a_1, b_1, c_1}{\alpha_3, \beta_3, \gamma_3} \\[2ex]
a_2\alpha_2 & \dfrac{a_2, b_2, c_2}{\alpha_1, \beta_1, \gamma_1} & \dfrac{a_2, b_2, c_2}{\alpha_3, \beta_3, \gamma_3} \\[2ex]
a_3\alpha_2 & \dfrac{a_3, b_3, c_3}{\alpha_1, \beta_1, \gamma_1} & \dfrac{a_3, b_3, c_3}{\alpha_3, \beta_3, \gamma_3}
\end{vmatrix}
$$

$$
+
\begin{vmatrix}
a_1\alpha_3 & \dfrac{a_1, b_1, c_1}{\alpha_1, \beta_1, \gamma_1} & \dfrac{a_1, b_1, c_1}{\alpha_2, \beta_2, \gamma_2} \\[2ex]
a_2\alpha_3 & \dfrac{a_2, b_2, c_2}{\alpha_1, \beta_1, \gamma_1} & \dfrac{a_2, b_2, c_2}{\alpha_2, \beta_2, \gamma_2} \\[2ex]
a_3\alpha_3 & \dfrac{a_3, b_3, c_3}{\alpha_1, \beta_1, \gamma_1} & \dfrac{a_3, b_3, c_3}{\alpha_2, \beta_2, \gamma_2}
\end{vmatrix}.
$$

Now by either of the interchanges

$$
\begin{pmatrix} a_1, & a_2, & a_3, & \alpha_1, & \alpha_2, & \alpha_3 \\ b_1, & b_2, & b_3, & \beta_1, & \beta_2, & \beta_3 \end{pmatrix}, \quad
\begin{pmatrix} a_1, & a_2, & a_3, & \alpha_1, & \alpha_2, & \alpha_3 \\ c_1, & c_2, & c_3, & \gamma_1, & \gamma_2, & \gamma_3 \end{pmatrix}
$$

the first columns of this,—and the first columns only,—would be affected, the a's and α's becoming b's and β's respectively in the one case, and c's and γ's in the other; and as neither interchange could affect the left-hand side of our identity, we should consequently note that thus three different expressions would be at once obtained for $|a_1b_2c_3| \cdot |\alpha_1\beta_2\gamma_3|$. Adding these together, and combining the nine determinants of the sum in sets of three by means of the addition-theorem (XLVII.), we should have finally

$$
3|a_1b_2c_3| \cdot |\alpha_1\beta_2\gamma_3| \;=\; 3
\begin{vmatrix}
\dfrac{a_1, b_1, c_1}{\alpha_1, \beta_1, \gamma_1} & \dfrac{a_1, b_1, c_1}{\alpha_2, \beta_2, \gamma_2} & \dfrac{a_1, b_1, c_1}{\alpha_3, \beta_3, \gamma_3} \\[2ex]
\dfrac{a_2, b_2, c_2}{\alpha_1, \beta_1, \gamma_1} & \dfrac{a_2, b_2, c_2}{\alpha_2, \beta_2, \gamma_2} & \dfrac{a_2, b_2, c_2}{\alpha_3, \beta_3, \gamma_3} \\[2ex]
\dfrac{a_3, b_3, c_3}{\alpha_1, \beta_1, \gamma_1} & \dfrac{a_3, b_3, c_3}{\alpha_2, \beta_2, \gamma_2} & \dfrac{a_3, b_3, c_3}{\alpha_3, \beta_3, \gamma_3}
\end{vmatrix},
$$

from which it is only necessary to delete the common factor 3.

JACOBI (1831–33).

[De transformatione integralis duplicis indefiniti

$$\int \frac{\partial\phi\partial\psi}{A + B\cos\phi + C\sin\phi + (A' + B'\cos\phi + C'\sin\phi)\cos\psi + (A'' + B''\cos\phi + C''\sin\phi)\sin\psi}$$

in formam simpliciorem $\int \frac{\partial\eta\partial\theta}{G - G'\cos\eta\cos\theta - G''\sin\eta\sin\theta}$.

[*Crelle's Journal*, viii. pp. 253–279, 321–357.]

[De transformatione et determinatione integralium duplicium commentatio tertia. *Crelle's Journal*, x. pp. 101–128.]

[De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas transformandis, quae solis quadratis variabilium constant; una cum variis theorematis de transformatione et determinatione integralium multiplicium. *Crelle's Journal*, xii. pp. 1–69.]

The first two of these memoirs may be viewed as continuations of a memoir with a similar title, which appeared in the second volume of *Crelle's Journal*, and to which we have already referred. They are noted here merely in order that the thread of investigation may be preserved unbroken, for the last memoir practically swallows up, by means of its splendid generalisations, all those that had gone before.

So long as we confine ourselves, in problems of transformation, to three independent variables, the explicit employment of the theory of determinants may be dispensed with. When, however, a sufficient number of special cases have been investigated, and an alluring glimpse has thereby been got of a generalisation involving them all, he who attempts the establishment of the generalisation must have recourse to the new weapon. In this latter position Jacobi now found himself. He wished to pass from the problem of orthogonal substitution in the case of three variables to the analogous problem in which the number of variables is n, or in his own words (p. 7):—

"Investigare substitutiones lineares huiusmodi

$$\begin{array}{l} y_1 = a_1'x_1 + a_2'x_2 + \ldots\ldots + a_n'x_n, \\ y_2 = a_1''x_1 + a_2''x_2 + \ldots\ldots + a_n''x_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = a_1^{(n)}x_1 + a_2^{(n)}x_2 + \ldots\ldots + a_n^{(n)}x_n, \end{array}$$

quibus efficiatur

$$y_1y_1 + y_2y_2 + \ldots\ldots + y_ny_n = x_1x_1 + x_2x_2 + \ldots\ldots + x_nx_n,$$

simulque data functio homogenea secundi ordinis variabilium $x_1, x_2, \ldots, x_n$ transformetur in aliam variabilium $y_1, y_2, \ldots, y_n$, de qua binarum producta evanuerunt."

This being the case he introduces determinants at the outset, fixing upon a notation which is practically Cauchy's, and immediately using properties of them without proof. Much that is contained in the memoir falls to be considered later, as it concerns special forms of determinants,—those afterwards known as Jacobians, axisymmetric determinants, and, of course, determinants of an orthogonal substitution. Indeed, the half-page of introduction is almost all that is of interest at present, but even in this a new and important theorem is enunciated. The first sentence of it stands as follows:—

"Supponamus, designantibus $a_k^{(m)}$ datas quantitates quaslibet, ex n æquationibus linearibus propositis huiusmodi

$$y_m = a_1^{(m)}x_1 + a_2^{(m)}x_2 + \ldots\ldots + a_n^{(m)}x_n,$$

per notas regulas resolutionis algebraicæ haberi æquationes formæ:

$$\mathrm{A}x_k = \beta_k'y_1 + \beta_k''y_2 + \ldots\ldots + \beta_k^{(n)}y_n.$$

Ipsum A supponimus denominatorem communem valorum incognitarum, qui per algorithmos notos formatur: sive fit

$$\mathrm{A} = \Sigma \pm a_1'a_2'' \ldots\ldots a_n^{(n)},$$

signo summatorio amplectente terminos omnes, qui indicibus aut inferioribus aut superioribus omnimodis permutatis proveniunt; signis eorum alternantibus secundum notam regulam, quam ita enunciare licet, ut termino cuilibet per certam permutationem *indicum* orto idem signum tribuatur, quo afficitur productum sequens conflatum e differentiis numerorum $1, 2, \ldots, n$

$$(2-1)(3-1)\ldots\ldots(n-1).(3-2)(4-2)\ldots\ldots(n-2).(4-3) \text{ etc.},$$

eadem *numerorum* permutatione facta."

It will be at once observed that Cauchy's italic letters *S*, *a*, *b* rre simply changed into Greek Σ, α, β.

The next sentence is :—

"Eadem notatione adhibita, sit

$$B = \Sigma \pm \beta_1{}'\beta_2{}'' \ldots . \beta_n{}^{(n)},$$

ubi ipsam B e quantitatibus $\beta_k{}^{(m)}$ eodem modo compositam accipimus, quo A ex ipsis $\alpha_k{}^{(m)}$ componitur. Quibus statutis observo fieri :

$$B = A^{n-1},$$

ac generalius :

$$\Sigma \pm \beta_1{}'\beta_2{}'' \ldots . \beta_m^{(m)} = A^{m-1}\Sigma \pm \alpha_{m+1}^{(m+1)}\alpha_{m+2}^{(m+2)} \ldots . \alpha_n^{(n)}." \quad \text{(xx. 3)}$$

As for the first theorem thus formulated, the credit of it is, of course, due to Cauchy: the second, however, is new, being indeed the theorem referred to above under Minding as having been foreshadowed by Lagrange, and left for over fifty years undisturbed. Jacobi evidently knew it in all its generality, for he adds—

"De qua formula generali cum pro variis valoribus ipsius *m*, tum indicibus et superioribus et inferioribus omnimodis permutatis, permultae aliae similes formulae profluunt."

The only other point to be noted at present is contained in the casual remark that the β's may be expressed as *differential coefficients* of A. When dealing later (p. 20), with a special form of determinant, he says—

"Data occasione observo generaliter, si $a_{\kappa,\lambda}$ et $a_{\lambda,\kappa}$ inter se diversi sunt, propositis *n* aequationibus linearibus hujusmodi :

$$\begin{array}{l} a_{1,1}u_1 + a_{1,2}u_2 + \ldots . + a_{1,n}u_n = v_1, \\ a_{2,1}u_1 + a_{2,2}u_2 + \ldots . + a_{2,n}u_n = v_2, \\ \ldots \ldots \ldots \ldots \ldots \\ a_{n,1}u_1 + a_{n,2}u_2 + \ldots . + a_{n,n}u_n = v_n, \end{array}$$

statuto

$$\Gamma = \Sigma \pm a_{1,1}a_{2,2} \ldots . a_{n,n}.$$

sequi vice versa

$$\Gamma u_1 = \frac{\partial\Gamma}{\partial a_{1,1}}v_1 + \frac{\partial\Gamma}{\partial a_{2,1}}v_2 + \ldots\ldots + \frac{\partial\Gamma}{\partial a_{n,1}}v_n,$$
$$\Gamma u_2 = \frac{\partial\Gamma}{\partial a_{1,2}}v_1 + \frac{\partial\Gamma}{\partial a_{2,2}}v_2 + \ldots\ldots + \frac{\partial\Gamma}{\partial a_{n,2}}v_n,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Gamma u_n = \frac{\partial\Gamma}{\partial a_{1,n}}v_1 + \frac{\partial\Gamma}{\partial a_{2,n}}v_2 + \ldots\ldots + \frac{\partial\Gamma}{\partial a_{n,n}}v_n. \qquad \text{(VI. 6)}$$

JACOBI (1835).

[De eliminatione variabilis e duabus aequationibus algebraicis. *Crelle's Journal*, XV. pp. 101–124.]

In a memoir having for its subject Bezout's method of eliminating x from the equations

$$\left.\begin{aligned} a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1 x + a_0 = 0, \\ b_n x^n + b_{n-1}x^{n-1} + \ldots\ldots + b_1 x + b_0 = 0, \end{aligned}\right\}$$

determinants are certain to occur explicitly or implicitly; and, the author being Jacobi, one is not surprised to find them introduced near the outset and employed thenceforward. It is of course only a special form of them which appears, viz., that afterwards distinguished by the term *persymmetric;* consequently, for the present the main contents of the memoir do not concern us. Note has to be made, however, of two points—(1) that while Jacobi does not discard his former notation $\Sigma \pm a_{r_0, s_0} a_{r_1, s_1} \ldots a_{r_m, s_m}$, he introduces and uses another, viz.,

$$a\begin{Bmatrix} r_0, r_1, r_2, \ldots, r_m \\ s_0, s_1, s_2, \ldots, s_m \end{Bmatrix}; \qquad \text{(VII. 8)}$$

(2) that a page is devoted to a fuller statement of the above-mentioned theorems regarding the adjugate determinant and a minor of the adjugate. The final sentence of this statement is all that need be reproduced. It is

"Sint igitur $r, r', r'', \ldots\ldots, r^{(n-1)}$ atque $s, s', s'', \ldots\ldots\ldots, s^{(n-1)}$ numeri omnes $0, 1, 2, \ldots, n-1$, quocunque ordine scripti; erit

$$A\left\{\begin{matrix} r^{(m)}, r^{(m+1)}, \ldots\ldots, r^{(n-1)} \\ s^{(m)}, s^{(m+1)}, \ldots\ldots, s^{(n-1)} \end{matrix}\right\} = L^{n-(1+m)}.a\left\{\begin{matrix} r, r', \ldots\ldots, r^{(m-1)} \\ s, s', \ldots\ldots, s^{(m-1)} \end{matrix}\right\}, \tag{xx. 4}$$

where L stands for $\Sigma \pm a_{0,0}a_{1,1} \ldots\ldots a_{n-1,n-1}$ and the adjugate of L is $\Sigma \pm A_{0,0}A_{1,1} \ldots A_{n-1,n-1}$. As before, no proofs of the theorems are given.

GRUNERT (1836).

[Supplemente zu Georg Simon Klügel's Wörterbuch der reinen Mathematik. Art. *Elimination* (I. Gleichungen des ersten Grades), ii. pp. 52-60.]

With Grunert it is necessary to take a long step backward. Although the memoirs of Bezout, Vandermonde, and Laplace were known to him, in addition to those of Hindenburg, Rothe, and Scherk, he advances only a short distance into the subject; his aim, indeed, is little more than the establishment of Cramer's rule for the solution of a set of simultaneous linear equations. His mode of presenting the subject, however, is fresh and interesting, the method of "undetermined multipliers" being taken to start with.

Writing his equations in the form

$$\left.\begin{array}{l} (1)_1x_1 + (2)_1x_2 + (3)_1x_3 + \ldots\ldots + (n)_1x_n = [1]_1 \\ (1)_2x_1 + (2)_2x_2 + (3)_2x_3 + \ldots\ldots + (n)_2x_n = [1]_2 \\ (1)_3x_1 + (2)_3x_2 + (3)_3x_3 + \ldots\ldots + (n)_3x_n = [1]_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (1)_nx_1 + (2)_nx_2 + (3)_nx_3 + \ldots\ldots + (n)_nx_n = [1]_n \end{array}\right\},$$

and taking $p_1, p_2, p_3, \ldots, p_n$ as multipliers, he readily shows of course that if the multipliers can be got to satisfy the conditions

$$\left.\begin{array}{l} (2)_1p_1 + (2)_2p_2 + (2)_3p_3 + \ldots\ldots + (2)_np_n = 0 \\ (3)_1p_1 + (3)_2p_2 + (3)_3p_3 + \ldots\ldots + (3)_np_n = 0 \\ (4)_1p_1 + (4)_2p_2 + (4)_3p_3 + \ldots\ldots + (4)_np_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (n)_1p_1 + (n)_2p_2 + (n)_3p_3 + \ldots\ldots + (n)_np_n = 0 \end{array}\right\},$$

$$\begin{aligned}
\Gamma u_1 &= \frac{\partial\Gamma}{\partial a_{1,1}}v_1 + \frac{\partial\Gamma}{\partial a_{2,1}}v_2 + \ldots\ldots + \frac{\partial\Gamma}{\partial a_{n,1}}v_n,\\
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&\cdots\cdots\cdots\cdots\cdots\cdots\\
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determinants are certain to occur explicitly or implicitly; and, the author being Jacobi, one is not surprised to find them introduced near the outset and employed thenceforward. It is of course only a special form of them which appears, viz., that afterwards distinguished by the term *persymmetric;* consequently, for the present the main contents of the memoir do not concern us. Note has to be made, however, of two points—(1) that while Jacobi does not discard his former notation $\Sigma \pm a_{r_0,s_0} a_{r_1,s_1} \ldots a_{r_m,s_m}$, he introduces and uses another, viz.,

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$$A\left\{\begin{matrix} r^{(m)}, r^{(m+1)}, \ldots\ldots, r^{(n-1)} \\ s^{(m)}, s^{(m+1)}, \ldots\ldots, s^{(n-1)} \end{matrix}\right\} = L^{n-(1+m)}.a\left\{\begin{matrix} r, r', \ldots\ldots, r^{(m-1)} \\ s, s', \ldots\ldots, s^{(m-1)} \end{matrix}\right\}, \qquad \text{(xx. 4)}$$

where L stands for $\Sigma \pm a_{0,0}a_{1,1} \ldots\ldots a_{n-1,n-1}$ and the adjugate of L is $\Sigma \pm A_{0,0}A_{1,1} \ldots A_{n-1,n-1}$. As before, no proofs of the theorems are given.

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Writing his equations in the form

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and taking $p_1, p_2, p_3, \ldots, p_n$ as multipliers, he readily shows of course that if the multipliers can be got to satisfy the conditions

$$\left.\begin{matrix} (2)_1p_1 + (2)_2p_2 + (2)_3p_3 + \ldots\ldots + (2)_np_n = 0 \\ (3)_1p_1 + (3)_2p_2 + (3)_3p_3 + \ldots\ldots + (3)_np_n = 0 \\ (4)_1p_1 + (4)_2p_2 + (4)_3p_3 + \ldots\ldots + (4)_np_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (n)_1p_1 + (n)_2p_2 + (n)_3p_3 + \ldots\ldots + (n)_np_n = 0 \end{matrix}\right\},$$

the value of x_1 will be

$$\frac{[1]_1 p_1 + [1]_2 p_2 + [1]_3 p_3 + \ldots\ldots + [1]_n p_n}{(1)_1 p_1 + (1)_2 p_2 + (1)_3 p_3 + \ldots\ldots + (1)_n p_n};$$

in other words, that x_1 can be determined at once if a function

$$(1)_1 p_1 + (1)_2 p_2 + (1)_3 p_3 + \ldots\ldots + (1)_n p_n$$

can be formed of such a character that it will vanish when instead of the coefficients $(1)_1$, $(1)_2$, $(1)_3$, . . . , $(1)_n$ we substitute the members of any one of the $n-1$ rows

$$\begin{array}{ccccc} (2)_1 & (2)_2 & (2)_3 & \ldots\ldots & (2)_n \\ (3)_1 & (3)_2 & (3)_3 & \ldots\ldots & (3)_n \\ (4)_1 & (4)_2 & (4)_3 & \ldots\ldots & (4)_n \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot & \cdot \\ (n)_1 & (n)_2 & (n)_3 & \ldots\ldots & (n)_n; \end{array}$$

the said function itself being the denominator of the value of x_1 and the numerator being derivable from the denominator by inserting $[1]_1$, $[1]_2$, $[1]_3$, . . . , $[1]_n$ in place of $(1)_1$, $(1)_2$, $(1)_3$, . . . ,$(1)_n$. Further, as any one of the unknowns may be made the first, the complete solution is thus put in prospect. "Alles kommt demnach auf die Entwickelung einer Function von der angegebenen Beschaffenheit an." (XIII. 5)

Two rules, Grunert says, have been given for the construction of such a function, one by Cramer, the other by Bezout. The former he states, and illustrates by constructing the desired function for the case where $n=4$. The proof of it is then attempted, and is said at the outset to consist essentially in establishing the proposition that a permutation and any other derivable from it by the simple interchange of two indices must, according to Cramer's rule, differ in sign. This proposition is therefore attacked. The permutation

$$\ldots\ldots (k)_\alpha \ldots\ldots\ldots (1)_{\alpha+\beta} \ldots\ldots \qquad \mathrm{A}$$

is taken in which the inferior indices are in their natural order 1, 2, 3, . . . , n, and k and 1 being interchanged, there arises the permutation

$$\ldots\ldots (1)_\alpha \ldots\ldots\ldots (k)_{\alpha+\beta} \ldots\ldots \qquad \mathrm{B}$$

The part preceding $(k)_\alpha$ in A is called I., which thus of course also denotes the part preceding $(1)_\alpha$ in B: the part between $(k)_\alpha$ and $(1)_{\alpha+\beta}$ in A or between $(1)_\alpha$ and $(k)_{\alpha+\beta}$ in B is called II.; and the remaining part common to both A and B is called III. The number of inversions in both, when 1 and k are left out of account, is denoted by γ: the number in both due to k and the division III. is denoted by λ: the number in A due to k and the division II. by λ': and the number in both due to the division I. and k by λ''. The counting of the inversions then takes place for the two permutations. In the case of A there are the inversions due

(1)	to I. and k,	which are	λ'' in number.
(2)	to I. and II.		
(3)	to I. and 1,	. . . .	$\alpha - 1$ . . .
(4)	to I. and III.		
(5)	to k and II.,	. . . .	λ' . . .
(6)	to k and 1,	. . . .	1 . . .
(7)	to k and III.,	. . . .	λ . . .
(8)	to II. and 1,	. . . .	$\beta - 1$ . . .
(9)	to II. and III.		
(10)	to 1 and III.,	. . . .	0 . . .

and as those not counted here are γ in number, the total is seen to be

$$\alpha + \beta + \gamma + \lambda + \lambda' + \lambda'' - 1.$$

Similarly in the case of B the total is found to be

$$\alpha + \beta + \gamma + \lambda - \lambda' + \lambda'' - 2.$$

But the former total exceeds the latter by $2\lambda' + 1$, and this being an odd number, the proposition is proved. (III. 26)

Before proceeding further it is important to note that Grunert here establishes a more definite theorem than he proposed to himself, viz., the theorem of Rothe (III. 7). If he attains greater simplicity it is in part due to the fact that instead of taking *any* two indices for interchange, k and r say, he takes k and 1.

To prove now that the function constructed in accordance with Cramer's rule will satisfy the requisite conditions, it suffices to show by means of this theorem that on making any one of the $n-1$ specified sets of substitutions the function will be transformed into one consisting of pairs of terms which annul each other; in other words, to prove Vandermonde's theorem regarding the effect of making two indices alike. This is done; and then it is shown how x_κ can be got by interchanging x_κ and x_1 in all the given equations, the first step being of course to establish the fact that the denominator of x_κ and the denominator of x_1 only differ in sign.

Bezout's rule of 1764 is next taken up, and shown to be identical in effect with Cramer's. The proof, by reason of the recurring character of the former, is inductive; that is to say, it is demonstrated that, if the two rules agree in the case of n unknowns, they must also agree in the case of $n+1$. Paraphrasing the proof, but taking for shortness' sake the case where $n=4$, we say that it is agreed that both rules give in this case the signed permutations

$$1234, \ -1243, \ +1423, \ -4123, \ -1324, \ +\ldots$$

Now for the case where $n=5$ Bezout's rule directs that to the end of each of these permutations, *e.g.*, the permutation -4123, a 5 is to be put, and asserts that the result -41235 will be one of the desired permutations with its proper sign. That it is a permutation of the first five integers is manifest, and since the number of inversions in 41235 is necessarily the same as the number in 4123, its sign is correct according to Cramer's rule. In order to obtain four other permutations, Bezout's rule then proceeds to bid us shift the 5 one place and alter the sign, shift the 5 another place and alter the sign again, and so on. The result is

$$+41253, \ -41523, \ +45123, \ -54123.$$

In regard to this, it is clear as before that permutations of the first five integers have been got, and that the altering of the sign simultaneously with the shifting of the 5 is in accordance with Cramer's rule, because every time that the 5 is moved one place to the left the number of inversions is increased by unity. The only

question remaining is as to whether *all* the permutations are thus obtainable; and as it is seen that each of the 24 permutations of the first four integers gives rise to 5 permutations of the first five, we have at once grounds for a satisfactory answer. (III. 27)

LEBESGUE (1837).

[Thèses de Mécanique et d'Astronomie. Première Partie: Formules pour la transformation des fonctions homogènes du second degré à plusieurs inconnues. *Liouville's Journal de Math.*, ii. pp. 337*–355.]

This simply-worded and clear exposition is a natural outcome of a study of Jacobi's memoirs on the subject. Like these it mainly concerns determinants of the special form afterwards individualised by the term axisymmetric; and, indeed, it is notable as being the first memoir in which a special name is given to a special form, the expression "déterminants symétriques" being repeatedly used for the particular determinants referred to.

His general definition is (p. 343):—

"Si l'on considère le système d'équations

$$\begin{cases} A_{1,1}t_1 + A_{1,2}t_2 + \ldots\ldots + A_{1,n}t_n = m_1, \\ A_{2,1}t_1 + A_{2,2}t_2 + \ldots\ldots + A_{2,n}t_n = m_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{n,1}t_1 + A_{n,2}t_2 + \ldots\ldots + A_{n,n}t_n = m_n, \end{cases}$$

le dénominateur commun des inconnues $t_1, t_2, \ldots, t_n$ est ce que l'on nomme le déterminant du système des nombres

$$(17) \qquad \begin{cases} A_{1,1} & A_{1,2} \ldots\ldots A_{1,n}, \\ A_{2,1} & A_{2,2} \ldots\ldots A_{2,n}, \\ \ldots & \ldots\ldots\ldots\ldots \\ A_{n,1} & A_{n,2} \ldots\ldots A_{n,n}. \end{cases}$$

Comme ce dénominateur peut changer de signe, selon le mode de solution qu'on emploiera, on conviendra de le

* *N.B.*—There are *two* pages numbered 337.

prendre de sorte que le terme $A_{1,1}A_{2,2}A_{3,3}\ldots A_{n,n}$ qui en fait partie, soit positif." (VIII. 3)

No use, however, is made of this for the purpose of establishing the properties of the functions, results being for the most part taken from previous investigators and merely restated. A notation for what are nowadays called the minors of a determinant is given in the following words (p. 344):— (XLI. 7)

"Ceci rappelé, si l'on représente par D le déterminant du système (17), par $[g, i]$ le déterminant du système qui se tire du système (17) par la suppression de la série horizontale de rang g et de la série verticale de rang i, et semblablement par la notation $\begin{bmatrix} g, & i \\ h, & k \end{bmatrix}$ le déterminant du système qui résulte de l'omission des séries horizontales de rangs g et i et des séries verticales de rangs i et k dans le système (17), on pourra, . . ."

REISS (1838).

[Essai analytique et géométrique. *Correspondance math. et phys.*, x. pp. 229–290.]

Reiss's memoir, the first part of which appeared in 1829, was never completed. In the course of some remarks introductory to the present essay, he says by way of excuse :—

"Je m'aperçus bientôt, et plusieurs savans me l'ont fait remarquer, que ces recherches, fussent-elles très-fécondes en résultats élégans, étaient trop abstraites pour intéresser le public qui n'apprécie les théories que selon le degré de leur utilité. J'ai donc tâché de montrer, par un exemple, de quelle manière on peut se servir de ces fonctions dans la géométrie analytique ; et j'ai choisi le *tétraèdre* qui, par le concours de plusieurs circonstances qu'on aura occasion de reconnaître plus tard, permettait une application très-facile et presque immédiate des premières conséquences auxquelles j'étais parvenu."

The analytical portion of the essay is to a considerable extent

identical with the original memoir. In so far as there is a difference, the change is towards greater simplicity, less seemingly aimless plunging into widely extensive theorems, and in general a better and more attractive style of exposition. Less space too is given to it,—not even half what is occupied by the portion on the tetrahedron, the main aim now being to urge on mathematicians the capabilities of the analysis in its application to geometry.

The matters falling to be noted as not having been given in the original memoir are few in number and of little importance. In restating the theorem

$$(abc\ .\ .\ .\ r,\ \ \overline{\alpha\beta\gamma\ .\ .\ .\ \rho}) = (\overline{abc\ .\ .\ .\ r},\ \ \alpha\beta\gamma\ .\ .\ .\ \rho)$$

the remark is incidentally made that the order of the terms on the one side is never the same as that on the other except when the number of bases is 1, 2, or 3; for example, the number of bases being 4, we have

$$(abcd,\ \overline{1234}) = a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_1b_3c_2d_4 \\ + a_1b_3c_4d_2 + \ .\ .\ .\ ,$$

whereas

$$(\overline{abcd},\ 1234) = a_1b_2c_3d_4 - a_1b_2d_3c_4 - a_1c_2b_3d_4 + \\ + a_1c_2d_3b_4 + \ .\ .\ .\ ,$$

the difference first appearing at the fourth term. (IX. 6)

Bezout's recurrent law of formation, formerly merely enunciated, is now accompanied by a demonstration. This is not without its weak point, the cause of which, as might be expected, is the awkwardness of Reiss's rule of signs. The first paragraph, which will suffice to show its character, is as follows (p. 233) :—

"Portons notre attention d'abord, seulement sur la fonction $(abc\ .\ .\ .\ r,\ \overline{\alpha\beta\gamma\ .\ .\ .\ \rho})$. Si l'on se représente la manière dont on fait les permutations des n élémens $\alpha, \beta, \gamma,\ .\ .\ .\ \rho$, on verra qu'à partir de la première, il y aura $1.2.3\ .\ .\ .\ (n-1)$ complexions qui commencent par α, et que, si l'on sépare cet élément par un trait vertical des autres, on aura à droite toutes les permutations des élémens $\beta, \gamma,\ .\ .\ .\ \rho$. Les $1.2.3.\ .\ .\ (n-1)$ premiers termes de $(abc\ .\ .\ .\ r,\ \ \overline{\alpha\beta\gamma\ .\ .\ .\ \rho})$ commencent donc tous par a^{α}, et puisque les signes de ces termes sont dé-

terminés d'après la manière exposée plus haut, on trouvera leur somme $= a^a(\dot{b}c \ldots r,\ \overline{\beta\gamma \ldots \rho})$."

Vandermonde's theorem regarding the effect, on the function, of interchanging two bases is stated generally, and a demonstration is given. The mode of demonstration, which occupies one page and a half, will be readily understood by seeing it applied in later notation to the case where there are *four* bases, that is to say, where the theorem to be proved is

$$|a_\alpha b_\beta c_\gamma d_\delta| = -|b_\alpha a_\beta c_\gamma d_\delta|.$$

By repeated use of the recurrent law of formation we have

$$\begin{aligned}
|a_\alpha b_\beta c_\gamma d_\delta| &= a_\alpha|b_\beta c_\gamma d_\delta| - a_\beta|b_\alpha c_\gamma d_\delta| + a_\gamma|b_\alpha c_\beta d_\delta| - a_\delta|b_\alpha c_\beta d_\gamma|,\\
&= a_\alpha\{\qquad b_\beta|c_\gamma d_\delta| - b_\gamma|c_\beta d_\delta| + b_\delta|c_\beta d_\gamma|\}\\
&\quad - a_\beta\{b_\alpha|c_\gamma d_\delta| \qquad - b_\gamma|c_\alpha d_\delta| + b_\delta|c_\alpha d_\gamma|\}\\
&\quad + a_\gamma\{b_\alpha|c_\beta d_\delta| - b_\beta|c_\alpha d_\delta| \qquad + b_\delta|c_\alpha d_\beta|\}\\
&\quad - a_\delta\{b_\alpha|c_\beta d_\gamma| - b_\beta|c_\alpha d_\gamma| + b_\gamma|c_\alpha d_\beta| \qquad\}.
\end{aligned}$$

By collecting the terms which have b_α for a common factor, b_β for a common factor, and so on, this result becomes

$$\begin{aligned}
|a_\alpha b_\beta c_\gamma d_\delta| &= -b_\alpha\{a_\beta|c_\gamma d_\delta| - a_\gamma|c_\beta d_\delta| + a_\delta|c_\beta d_\gamma|\}\\
&\quad + b_\beta\{a_\alpha|c_\gamma d_\delta| - a_\gamma|c_\alpha d_\delta| + a_\delta|c_\alpha d_\gamma|\}\\
&\quad - b_\gamma\{a_\alpha|c_\beta d_\delta| - a_\beta|c_\alpha d_\delta| + a_\delta|c_\alpha d_\beta|\}\\
&\quad + b_\delta\{a_\alpha|c_\beta d_\gamma| - a_\beta|c_\alpha d_\gamma| + a_\gamma|c_\alpha d_\beta|\},\\
&= -b_\alpha|a_\beta c_\gamma d_\delta| + b_\beta|a_\alpha b_\gamma d_\delta| - b_\gamma|a_\alpha c_\beta d_\delta| + b_\delta|a_\alpha c_\beta d_\gamma|,\\
&= -|b_\alpha a_\beta c_\gamma d_\delta|,
\end{aligned}$$

as was to be proved. (XI. 5)

The suggestion readily arises that this process would be equally applicable in proving Vandermonde's theorem regarding the vanishing of a function in which two bases are identical, and the process, it may be remembered, was actually so employed by Desnanot.

One of the theorems given by Scherk, and later by Drinkwater, appears in the following form (p. 240), the peculiar notation adopted for a determinant with a row of unit elements being constantly employed throughout the remainder of the essay:—

"Si une des bases, par exemple a, est telle que la quantité qu'elle représente soit la même quel que soit l'exposant dont elle est affectée, c'est-à-dire, si $a^\alpha = a^\beta = a^\gamma = \ . \ . \ .$, on aura

$$(abc \ . \ . \ . \ r, \quad \alpha\beta\gamma \ . \ . \ . \ \rho)$$

$$= a^a[(bc\ldots r, \beta\gamma\ldots\rho) - (bc\ldots r, \alpha\gamma\ldots\rho) + (bc\ldots r, \alpha\beta\delta\ldots\rho) \mp \ldots].$$

La quantité qui se trouve sous la parenthèse, peut donc être représentée de la manière suivante :

$$(\mathrm{I}bc \ . \ . \ . \ r, \quad \alpha\beta\gamma \ . \ . \ . \ \rho); \qquad \text{(XLVIII. 3)}$$

en admettant une fois pour toutes que le chiffre romain I soit tel que $1 = \mathrm{I}^\alpha = \mathrm{I}^\beta = \mathrm{I}^\gamma = \ . \ . \ .$ Il va sans dire que toutes les propriétés qui ont lieu pour $(abc \ . \ . \ . \ r, \alpha\beta\gamma \ . \ . \ . \ \rho)$ se rapportent également à

$$(\mathrm{I}bc \ . \ . \ . \ r, \quad \alpha\beta\gamma \ . \ . \ . \ \rho)."$$

The character of the identities used in the treatment of the tetrahedron will be learned from a glance at the following examples:—

$$a_1(\mathrm{I}bc, 123) - b_1(\mathrm{I}ac, 123) + c_1(\mathrm{I}ab, 123) = (abc, 123).$$

$$(a_1 - a_2)(\mathrm{I}bc, 123) - (b_1 - b_2)(\mathrm{I}ac, 123) + (c_1 - c_2)(\mathrm{I}ab, 123) = 0.$$

$$\begin{aligned}(ab, 12)(ac, 34) - (ab, 34)(ac, 12) &= -a_1(abc, 234) + a_2(abc, 134), \\ &= +a_3(abc, 124) - a_4(abc, 123).\end{aligned}$$

$$(\mathrm{I}ab, 123)(\mathrm{I}ac, 124) - (\mathrm{I}ab, 124)(\mathrm{I}ac, 123) = -(a_1 - a_2)(\mathrm{I}abc, 1234).$$

$$(\mathrm{I}ab, 123)(abc, 124) - (\mathrm{I}ab, 124)(abc, 123) = +(ab, 12)(\mathrm{I}abc, 1234).$$

The first of these we have already seen used by Minding; the second is nothing more than the manifest identity,

$$\begin{vmatrix} . & 1 & 1 & 1 \\ a_1 - a_2 & a_1 & a_2 & a_3 \\ b_1 - b_2 & b_1 & b_2 & b_3 \\ c_1 - c_2 & c_1 & c_2 & c_3 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_1 & a_2 & a_3 \\ b_1 & b_1 & b_2 & b_3 \\ c_1 & c_1 & c_2 & c_3 \end{vmatrix} = 0;$$

the third is evidently the equatement of two expansions of

$$\begin{vmatrix} a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_3 & \cdot & \cdot & a_4 \\ a_3 & a_1 & a_2 & a_4 \\ b_3 & b_1 & b_2 & b_4 \\ c_3 & c_1 & c_2 & c_4 \end{vmatrix};$$

the fourth is a case of the fifth: and the fifth is itself a case of a theorem (C') of Desnanot's.

CATALAN (1839).

[Sur la transformation des variables dans les intégrales multiples. *Mémoires couronnés par l'Académie royale . . . de Bruxelles*, xiv. 2^{me} partie, 49 pp.]

The first of the four parts into which Catalan's memoir is divided bears the title "*Valeurs générales des inconnues dans les équations du premier degré, et propriétés des dénominateurs communs,*" and in the introduction it is said to contain several remarkable new properties of the functions called *resultants* by Laplace "et connues aujourd'hui sous le nom de *déterminants.*"

His method of dealing with the opening problem is to derive the solution of n equations with n unknowns from the solution of $n-1$ equations with $n-1$ unknowns; or more definitely, to show that if the multipliers $\lambda_1, \lambda_2, \lambda_3$ necessary for the solution of the set of equations,

$$\left.\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 &= \alpha_1 \\ a_2x_1 + b_2x_2 + c_2x_3 &= \alpha_2 \\ a_3x_1 + b_3x_2 + c_3x_3 &= \alpha_3 \end{aligned}\right\},$$

be the determinants of the systems

$$\begin{matrix} a_2 & b_2 \\ a_3 & b_3, \end{matrix} \qquad \begin{matrix} a_3 & b_3 \\ a_1 & b_1, \end{matrix} \qquad \begin{matrix} a_1 & b_1 \\ a_2 & b_2, \end{matrix}$$

then the multipliers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ necessary for the solution of the set

$$\left.\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 &= \alpha_1 \\ a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4 &= \alpha_2 \\ a_3x_1 + b_3x_2 + c_3x_3 + d_3x_4 &= \alpha_3 \\ a_4x_1 + b_4x_2 + c_4x_3 + d_4x_4 &= \alpha_4 \end{aligned}\right\}$$

are the determinants of the systems

$$\begin{matrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{matrix} \qquad \begin{matrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \end{matrix} \qquad \begin{matrix} a_4 & b_4 & c_4 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \qquad \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} . \tag{XIII. 6}$$

This of course means that in the first case

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = 0 ,$$
$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = 0 ,$$
$$\text{and} \qquad x_3 = \frac{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3}{\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3} ;$$

and in the other

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 = 0 ,^*$$
$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_4 = 0 ,$$
$$c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 + c_4\lambda_4 = 0 ,$$
$$\text{and} \qquad x_4 = \frac{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4}{\lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 + \lambda_4 d_4} .$$

The proof is disappointingly weak and unsatisfactory, and, what is still more surprising, rests at one point on a manifest inaccuracy. He says (p. 9)—

> "Par un calcul direct, on vérifie la formule (6) et les relations (5) pour le cas de trois équations. En même temps, l'on reconnait que
>
> "1° Le dénominateur de la valeur de x_3, par exemple, renferme toutes les combinaisons trois à trois des coefficients, chaque combinaison ne contenant ni deux fois la même lettre, ni deux fois le même indice.
>
> "2° Deux termes qui, dans l'expression de ce dénominateur, peuvent se déduire l'un de l'autre par une permutation tournante ont même signe.
>
> "3° Deux termes qui ne diffèrent que par le changement d'une lettre en une autre, et réciproquement, sont de signes contraires.
>
> "4° Par suite, le dénominateur est le même pour toutes les

* Note, however, the error in sign of λ_2 and λ_4.

inconnues, pourvu que l'on prenne convenablement le signe du numérateur."

He then proceeds—

"Supposons donc que pareille vérification ait été faite pour $n-1$ équations entre $n-1$ inconnues, je dis qu'elle se fera encore dans le cas de n équations."

Now although the statement in 2° is true for the case of three equations, it is not true generally, and therefore cannot be proved.*

The theorems which follow this introductory matter concern a special determinant, viz., the determinant of the system,

$$\begin{array}{cccccc} a_1 & b_1 & c_1 & \cdots\cdots & k_1 & l_1 \\ a_2 & b_2 & c_2 & \cdots\cdots & k_2 & l_2 \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \\ a_n & b_n & c_n & \cdots\cdot & k_n & l_n , \end{array}$$

in which the elements are connected by the $\frac{1}{2}n(n-1)$ relations

$$\left.\begin{array}{l} a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n = 0 \\ a_1c_1 + a_2c_2 + a_3c_3 + \ldots + a_nc_n = 0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ a_1l_1 + a_2l_2 + a_3l_3 + \ldots + a_nl_n = 0 \\ b_1c_1 + b_2c_2 + b_3c_3 + \ldots + b_nc_n = 0 \\ b_1d_1 + b_2d_2 + b_3d_3 + \ldots + b_nd_n = 0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ b_1l_1 + b_2l_2 + b_3l_3 + \ldots + b_nl_n = 0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ k_1l_1 + k_2l_2 + k_3l_3 + \ldots + k_nl_n = 0 \end{array}\right\} .$$

Such determinants are only a little less special than determinants of an orthogonal substitution, and thus naturally fall to be considered later along with those of the latter class.

* In the proof he is fortunate (or unfortunate) enough to use another special case in which the statement is true. He says:—"Les deux termes $a_7b_6c_1d_3e_5f_2$ et $e_7f_6a_1b_3c_5d_2$ qui entrent dans D_4, et qui se déduisent l'un de l'autre par une permutation tournante entre les lettres ont même signe."

GARNIER (1814).

[Analyse Algébrique, faisant suite à la première section de l'algèbre. 2^e édition, revue et considérablement augmentée. xvi. + 668 pp. Paris.]

The title of Garnier's chapter xxvii. (pp. 541–555) is "*Développement de la théorie donnée par M. Laplace pour l'élimination au premier degré.*" It consists, however, of nothing but a simple exposition, confessedly borrowed from Gergonne's paper of 1813, and six pages of extracts from Laplace's original memoir of 1772. As forming part of a popular text-book, it probably did more service in bringing the theory to the notice of mathematicians than a memoir in a recondite serial publication could have done; and we certainly know that Sylvester, who afterwards did so much to advance the theory, expresses himself indebted to it.

SYLVESTER (1839).

[On Derivation of Coexistence: Part I.* Being the Theory of simultaneous simple homogeneous Equations. *Philosophical Magazine*, xvi. pp. 37–43.]

Sylvester was apparently first brought into contact with determinants while investigating the subject of the elimination of x between two equations of the m^{th} and n^{th} degrees. At the close of a paper on this subject (*Phil. Mag.*, xv. p. 435) he says—"I trust to be able to present the readers of this magazine with a *direct* and *symmetrical* method of eliminating any number of unknown quantities between any number of equations of any degree, by a newly invented process of symbolical multiplication, and the use of *compound* symbols of notation." These last words, indicative of the method, exactly describe the matter dealt with in the paper we have now come to, and as will soon be seen, the functions which are the outcome of the said "compound symbol" of operations are determinants.

It would also appear that Sylvester was unacquainted with any

* Misprint for II., as an expression in the paper itself shows.

of the important memoirs of his predecessors regarding the functions: the twenty-seventh chapter of Garnier's *Analyse Algébrique*, to which he refers, may very probably indicate the extent of his knowledge.

Premising that he is going to use such symbols as $a_1, a_2, \ldots$ he calls the letter a the "base," and the complete symbol "an argument of the base," a_1 being the first argument, a_2 the second, and so on. Taking then a number of expressions, "each of which is made up of one or more terms, consisting solely of linear arguments of different bases, *i.e.*, characters bearing indices below but none above," *e.g.*, the expressions,

$$a_1 - b_1, \quad a_1 - c_1;$$

he alters them by writing the index-numbers *above*, *e.g.*,

$$a^1 - b^1, \quad a^1 - c^1;$$

takes the product of these resulting expressions in its expanded form

$$a^2 - a^1b^1 - a^1c^1 + b^1c^1;$$

and then reverses the operation on the index-numbers, thus finally obtaining

$$a_2 - a_1b_1 - a_1c_1 + b_1c_1.$$

The full series of these operations he indicates by the letter ζ, and denotes by the name of "*zeta-ic multiplication*." Thus, as results in zeta-ic multiplication, we have

$$\zeta(a_1 - b_1)(a_1 - c_1) = a_2 - a_1b_1 - a_1c_1 + b_1c_1,$$

and
$$\zeta(a_1 + b_1)^2 = a_2 + 2a_1b_1 + b_2.*$$

Further ζ_{+r} is used to denote that, after the operations ζ have been performed, the indices are all to be increased by r, the result of so doing being called the zeta-ic product *in its* r^{th} *phase.*

He nexts recalls a notation previously introduced by him for the functions which came later to be known shortly as difference-products; denoting, for example,

* He would not even hesitate to extend the use of the symbol, denoting, for example,

$$1 - \frac{a_2}{1.2} + \frac{a_4}{1.2.3.4} - \ldots \text{ by } \zeta\cos(a_1).$$

$$(b-a)(c-a)(c-b) \text{ by } \mathrm{PD}(abc),$$

$$(b-a)(c-a)(c-b)(d-a)(d-b)(d-c) \text{ by } \mathrm{PD}(abcd),$$

and $$\therefore \quad abc(b-a)(c-a)(c-b) \text{ by } \mathrm{PD}(0abc).$$

Lastly, he combines the two notations; and any reader who remembers Cauchy's mode of solving a set of simultaneous linear equations can with certainty predict the result of the union to be determinants. A new notation and a new name for the functions thus come into being together, the determinant of the system

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}$$

being represented by

$$\zeta abc\mathrm{PD}(abc) \quad \text{or} \quad \zeta\mathrm{PD}(0abc), \qquad (\text{VII. } 9)$$

and being called a *zeta-ic product of differences.* (XV. 7)

These special zeta-ic products being reached, the rest of the paper is taken up with an account of some of their properties, and the application of them to the discussion of simultaneous linear equations. Some of the matter may be passed over as being already familiar to us, although its earlier appearances were certainly made in a less picturesque dress. The first really fresh theorem concerns the zeta-ic multiplication of a determinant $\zeta\mathrm{PD}\,(0abc\ldots l)$ by those symmetric functions of $a, b, c, \ldots, l$, which we should now denote by

$$\Sigma a, \ \Sigma ab, \ \Sigma abc, \ \ldots .$$

but which Sylvester writes in the form

$$\mathrm{S}_1(abc\ldots l), \quad \mathrm{S}_2(abc\ldots l), \quad \mathrm{S}_3(abc\ldots l), \quad \ldots .$$

In his own words it stands as follows (p. 39):—

"Let $a, b, c, \ldots l$, denote any number of independent bases, say $(n-1)$; but let the argument of each base be periodic, and the number of terms in each period the same for every base, namely (n), so that

$$\begin{array}{ll} a_r = a_{r+n} = a_{r-n} & a_n = a_0 = a_{-n} \\ b_r = b_{r+n} = b_{r-n} & b_n = b_0 = b_{-n} \\ c_r = c_{r+n} = c_{r-n} & c_n = c_0 = c_{-n} \\ \cdots\cdots & \cdots\cdots \\ l_r = l_{r+n} = l_{r-n} & l_n = l_0 = l_{-n}, \end{array}$$

r being any number whatever. Then

$$\begin{aligned} \zeta_{-1}\mathrm{PD}(0abc\ldots l) &= \zeta\big(\mathrm{S}_1(abc\ldots l).\zeta\mathrm{PD}(0abc\ldots l)\big) \\ \zeta^{-2}\mathrm{PD}(0abc\ldots l) &= \zeta\big(\mathrm{S}_2(abc\ldots l).\zeta\mathrm{PD}(0abc\ldots l)\big) \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ \zeta_{-r}\mathrm{PD}(0abc\ldots l) &= \zeta\big(\mathrm{S}_r(abc\ldots l).\zeta\mathrm{PD}(0abc\ldots l)\big)." \end{aligned}$$

The limitation made upon the arguments of the base would seem to imply that the theorem only concerned determinants of a very special kind. Such, however, is not the case. A special example in more modern notation will bring out its true character. Let the determinant chosen be

$$| a_1 b_2 c_3 d_4 |,$$

and let the symmetric function be

$$ab + ac + ad + bc + bd + cd.$$

Multiplying the two together "zeta-ically," that is to say, in accordance with the law

$$a_r \times a_s = a_{r+s},$$

we find that 120 of the total 144 terms of the product cancel each other, and that the remaining 24 terms constitute the determinant

$$| a_1 b_2 c_4 d_5 |,$$

the identity thus reached being

$$\zeta\big(| a_1 b_2 c_3 d_4 | . \Sigma ab\big) = | a_1 b_2 c_4 d_5 |.$$

Now Sylvester's ζPD notation being unequal to the representation of the determinant $| a_1 b_2 c_4 d_5 |$ in which the index-numbers do not proceed by the common difference 1, he would seem to have been compelled to give a periodic character to the arguments of the

bases in order to remove the difficulty. At any rate the difficulty *is* removed; for the number of terms in the period being 5 the index-numbers 4 and 5 become changeable into -1 and 0, and thus we can have

$$|a_1b_2c_4d_5| = |a_1b_2c_{-1}d_0|,$$
$$= |a_{-1}b_0c_1d_2|,$$

—a determinant in which the index-numbers proceed by the common difference 1, and which is obtainable from $|a_1b_2c_3d_4|$ by diminishing each index-number by 2. Sylvester's form of the result thus is

$$\zeta\{S_2(abcd).\zeta PD(0abcd)\} = \zeta_{-2}(0abcd).*$$

Following this comes the application to simultaneous linear equations, or as they are called "equations of coexistence." The system is represented by the typical equation

$$a_rx + b_ry + c_rz + \ldots + l_rt = 0,$$

in which r can take up all integer values from $-\infty$ to $+\infty$, there being really, however, only n equations, because of the periodicity imposed on the arguments of the bases. One so-called "leading theorem" is given in regard to the system, its subject being the derivation of an equation linear in $x, y, z, \ldots, t$ by a combination of the equations of the system. The theorem is enunciated as follows (p. 40):—

> "Take $f, g, \ldots, k$ as the *arbitrary* bases of new and absolutely independent but periodic arguments, having the same

* It is rather curious that Sylvester overlooks the fact that the legitimate equatement of two zeta-ic products implies an identity altogether independent of the existence of zeta-ic multiplication. Thus, the identity just discussed is essentially the same as the identity

$$\begin{vmatrix} a & a^2 & a^3 & a^4 \\ b & b^2 & b^3 & b^4 \\ c & c^2 & c^3 & c^4 \\ d & d^2 & d^3 & d^4 \end{vmatrix} \times (ab + ac + ad + bc + bd + cd) = \begin{vmatrix} a & a^2 & a^4 & a^5 \\ b & b^2 & b^4 & b^5 \\ c & c^2 & c^4 & c^5 \\ d & d^2 & d^4 & d^5 \end{vmatrix},$$

where the index-number denotes a power and the multiplication is performed in accordance with the ordinary algebraic laws. From this point of view the above quoted proposition of Sylvester's involves an important theorem regarding the special determinants afterwards known by the name of *alternants*.

index of periodicity (n) as $a, b, c, \ldots, l$, and being in number $(n-1)$; *i.e.*, one fewer than there are units in that index.

"The number of *differing* arbitrary constants thus *manufactured* is $n(n-1)$.

"Let $Ax + By + Cz + \ldots + Lt = 0$ be the general *prime* derivative from the given equations, then we may make

$$\begin{aligned} A &= \zeta PD(0afg \ \ldots \ k) \\ B &= \zeta PD(0bfg \ \ldots \ k) \\ C &= \zeta PD(0cfg \ \ldots \ k) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ L &= \zeta PD(0lfg \ \ldots \ k)\text{''} \end{aligned} \qquad \text{(XIII. 7)}$$

As in the case of the other theorems, no demonstration is vouchsafed. In order, however, that the connection between it and previous work may be more readily manifest, it is desirable to indicate how it would most probably be established now. Taking the case where the number of unknowns is *three* and the number of given equations *four*, viz.—

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \\ a_4x + b_4y + c_4z = 0 \end{aligned}\right\},$$

we should form an array of $4(4-1)$, *i.e.* 12, arbitrary quantities,

$$\begin{matrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \\ f_4 & g_4 & h_4, \end{matrix}$$

from which we should select the multiplier $|f_2g_3h_4|$ for the first given equation, the multiplier $|f_1g_3h_4|$ for the second equation, and so on. The multiplication then being performed we should by addition obtain

$$|a_1f_2g_3h_4|\, x + |b_1f_2g_3h_4|\, y + |c_1f_2g_3h_4|\, z = 0,$$

which is what Sylvester would call "the general prime derivative of

the four given equations," the process being an instance of what he would similarly term the "derivation of coexistence."

By proper choice of the arbitrary quantities it may be readily shown, as Sylvester proceeds to do, that the theorem gives (1) the result of the elimination of n unknowns from n equations; (2) the *two* equations of condition in the case of $n+1$ equations connecting n unknowns; (3) the ratio of any two unknowns in the case of $n-1$ equations connecting n unknowns; and (4) the relation between any three unknowns in the case of $n-2$ equations connecting n unknowns. For example, the equations being

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}\right\},$$

the theorem gives the general derivative

$$\begin{vmatrix} a_1 & f_1 & g_1 \\ a_2 & f_2 & g_2 \\ a_3 & f_3 & g_3 \end{vmatrix} x + \begin{vmatrix} b_1 & f_1 & g_1 \\ b_2 & f_2 & g_2 \\ b_3 & f_3 & g_3 \end{vmatrix} y + \begin{vmatrix} c_1 & f_1 & g_1 \\ c_2 & f_2 & g_2 \\ c_3 & f_3 & g_3 \end{vmatrix} z = 0,$$

which is true whatever $f_1, f_2, f_3, g_1, g_2, g_3$ may be. By putting $f_1, f_2, f_3, g_1, g_2, g_3 = b_1, b_2, b_3, c_1, c_2, c_3$, this takes the form

$$|a_1b_2c_3|x + |b_1b_2c_3|y + |c_1b_2c_3|z = 0,$$

whence the equation of condition, or resultant of elimination,

$$|a_1b_2c_3| = 0.$$

As a corollary to one of the deductions from the leading theorem, —the deduction numbered (3) above,—the following proposition of a different character is given (p. 42):—

"If there be any number of bases $(abc \ldots l)$, and any other, two fewer in number, $(fg \ldots k)$,

$$\begin{aligned} &\zeta\mathrm{PD}(afg \ldots k) \times \zeta\mathrm{PD}(bc \ldots l) \\ +&\zeta\mathrm{PD}(bfg \ldots k) \times \zeta\mathrm{PD}(ac \ldots l) \\ +&\zeta\mathrm{PD}(afg \ldots k) \times \zeta\mathrm{PD}(bc \ldots l) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ +&\zeta\mathrm{PD}(lfg \ldots k) \times \zeta\mathrm{PD}(abc \ldots \ ) = 0, \end{aligned}$$

a formula that from its very nature suggests and proves a wide extension of itself." (xxIII. 10)

It belongs evidently to the class of vanishing aggregates of products of pairs of determinants, of which so many instances have presented themselves. There is a manifest misprint in the third product, which should surely be

$$\zeta\text{PD}(cfg\ .\ .\ .\ k) \times \zeta\text{PD}(ab\ .\ .\ .\ l);$$

and there is an error in the signs connecting the products, which, instead of being all +, should be + and − alternately. When the determinants involved are of the third order, the theorem in the later notation is

$$|a_1f_2g_3|\cdot|b_1c_2d_3| - |b_1f_2g_3|\cdot|a_1c_2d_3| + |c_1f_2g_3|\cdot|a_1b_2d_3| - |d_1f_2g_3|\cdot|a_1b_2c_3| = 0,$$

which is readily recognised as an identity given by Bezout.

With this theorem the paper proper ends, but in a postscript an additional theorem of a curious character is given. As enunciated by the author—even his double mark of exclamation being reprinted—it is (p. 43):—

"Let there be $(n-1)$ bases $a, b, c, \ldots, l$, and let the arguments of each be "recurrents of the n^{th} order," that is to say, let

$$a_i = \phi\left(\cos.\frac{2\pi i}{n}\right), \quad b_i = \psi\left(\cos.\frac{2\pi i}{n}\right), \quad c_i = \chi\left(\cos.\frac{2\pi i}{n}\right),$$

$$\ldots\ldots, \quad l_i = \omega\left(\cos.\frac{2\pi i}{n}\right).$$

Let R_r denote that any symmetrical function of the r^{th} degree is to be taken of the quantities in a parenthesis which come after it, and let ϑ indicate any function whatever. Then the zeta-ic product,

$$\zeta(\zeta\text{R}_r(abc\ .\ .\ .\ l) \times \zeta_\epsilon\vartheta\text{PD}(0abc\ .\ .\ .\ l))$$

is equal to the product of the *number*

$$R_t\Big(\Big(\cos.\frac{2\pi}{n}+\sqrt{-1}.\sin\frac{2\pi}{n}\Big),\Big(\cos.\frac{4\pi}{n}+\sqrt{-1}.\sin\frac{4\pi}{n}\Big)$$

$$\Big(\cos.\frac{6\pi}{n}+\sqrt{-1}.\sin\frac{6\pi}{n}\Big)\ldots\ldots$$

$$\cos.\Big(\frac{(2n-1)\pi}{n}+\sqrt{-1}.\sin\frac{2(n-1)\pi}{n}\Big)\Big)$$

multiplied by the zeta-ic phase

$$\zeta_{\varrho-t}\,\vartheta\mathrm{PD}(0abc\ldots l)\ !!\text{"}$$

Unfortunately the meaning of the proposition is seriously obscured by misprints and inaccurate use of symbols. Instead of "r^{th}" degree we should have t^{th} degree; ζ preceding $R_t\,(abc\ldots l)$ is meaningless, and should be deleted; ζ preceding ϑPD $(0abc\ldots l)$ in the first member of the identity is unnecessary when a ζ has already been printed at the commencement; and the subscript $_\varrho$, although giving an appearance of greater generality, serves no purpose whatever. Making the corrections thus suggested, and denoting

$$\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n},\quad \cos\frac{4\pi}{n}+\sqrt{-1}\sin\frac{4\pi}{n},\quad \ldots\ldots,$$

which are the roots of the equation

$$x^{n-1}+x^{n-2}+x^{n-3}+\ldots+x+1=0,$$

by $\alpha,\beta,\gamma,\ldots,\lambda$, we are enabled to put the theorem in the more elegant form

$$\zeta\{R_t(a,b,c\ \ldots,l).\vartheta.\mathrm{PD}(0,a,b,c,\ldots,l)\}$$
$$=\zeta_{-t}\{R_t(\alpha,\beta,\gamma,\ldots,\lambda).\vartheta.\mathrm{PD}(0,a,b,c,\ldots,l).$$

It is readily seen to be a generalisation of the first theorem of the paper, into which it degenerates when ϑ, instead of being any function of $a,b,c,\ldots l$, is a constant, and R_t, instead of being *any* symmetric function, is one of the series Σa, Σab, Σabc, ... As, however, the constant $R_t(\alpha,\beta,\gamma,\ldots\lambda)$ on the right-hand side will then be one of the series $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, ... and will not therefore be $+1$ unless when t is even, there must be an inattention to sign in one or other theorem. The matter can be more appropriately inquired into when we come to the subject of alternants, because, as has been pointed out in a recent footnote, it is to this

branch of the subject that identities between two zeta-ic multiplications of difference-products really belong.

This early paper, one cannot but observe, has all the characteristics afterwards so familiar to readers of Sylvester's writings,—fervid imagination, vigorous originality, bold exuberance of diction, hasty if not contemptuous disregard of historical research, the outstripping of demonstration by enunciation, and an infective enthusiasm as to the vistas opened up by his work.

SYLVESTER (1840).

[A method of determining by mere inspection the derivatives from two equations of any degree. *Philosophical Magazine*, xvi. pp. 132–135.]

The two equations taken are

$$\left.\begin{array}{l} a_n x^n + a_{n-1}x^{n-1} + \ . \ . \ . \ + a_1 x + a_0 = 0 \\ b_n x^n + b_{n-1}x^{n-1} + \ . \ . \ . \ + b_1 x + b_0 = 0 \end{array}\right\} ,$$

and rules are given for attaining three different objects, viz. (1) a rule for absolutely eliminating x; (2) a rule for finding the prime derivative of the first degree, that is to say of the form $\mathrm{A}x - \mathrm{B} = 0$; (3) a rule for finding the prime derivative of any degree. The first of these concerns the process afterwards so well known by the name "dialytic." Only part of it need be given (p. 132):—

"Form out of the a progression of coefficients m lines, and in like manner out of the b progression of coefficients form n lines in the following manner: Attach $m-1$ zeros all to the *right* of the terms in the a progression; next attach $m-2$ zeros to the right and carry 1 over to the left; next attach $m-3$ zeros to the right and carry 2 over to the left. Proceed in like manner until all the $m-1$ zeros are carried over to the left, and none remain on the right. The m lines thus formed are to be written under one another.

Proceed in like manner to form n lines out of the b progression by scattering $n-1$ zeros between the right and left.

If we write these n lines under the m lines last obtained, we

shall have a solid square $m+n$ terms deep and $m+n$ terms broad." (LIV. 1)

The rest of the rule deals of course with the formation of the terms from this square of elements, the old and familiar method being followed of taking all possible permutations and separating the permutations into positive and negative. As applied by Sylvester in the case of the elimination of x between the equations

$$\left.\begin{aligned} ax^2 + bx + c &= 0 \\ lx^2 + mx + n &= 0 \end{aligned}\right\},$$

that is to say, as applied to the development of the determinant of the system

$$\begin{matrix} a & b & c & 0 \\ 0 & a & b & c \\ l & m & n & 0 \\ 0 & l & m & n, \end{matrix}$$

the method is lengthy.

No hint at an explanation of this or either of the two other rules is given. The principle at the basis of them all, however, is essentially that of the preceding paper. A single example will make this plain, and will at the same time serve to give a better idea of the two remaining rules than could be got by mere quotation.* Let the two given equations be

$$\left.\begin{aligned} ax^3 + bx^2 + cx + d &= 0 \\ \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon &= 0 \end{aligned}\right\},$$

and suppose that it is desired to obtain their "prime derivative" of the 2nd (r^{th}) degree, that is to say, the derivative of the form

$$\mathrm{A}x^2 + \mathrm{B}x + \mathrm{C} = 0.$$

Taking the first equation followed by $m-r-1$ equations derived from it by repeated multiplication by x, and then the second equation followed by $n-r-1$ equations derived from it in like manner, we have $m+n-2r$ equations,

$$\left.\begin{aligned} ax^3 + bx^2 + cx + d &= 0 \\ ax^4 + bx^3 + cx^2 + dx \quad &= 0 \\ \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon &= 0 \end{aligned}\right\},$$

* The third rule is incorrectly stated.

from which we have to deduce an equation involving no power of x higher than the 2nd. To do so we employ, as just stated, exactly the same method as was used in obtaining the "leading theorem" of the preceding paper. That is to say, we form multipliers

$$\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix}, \quad -\begin{vmatrix} . & a \\ \alpha & \beta \end{vmatrix}, \quad \begin{vmatrix} . & a \\ a & b \end{vmatrix},$$

effect the multiplications, and add, the result being

$$\begin{vmatrix} . & a & b \\ a & b & c \\ \alpha & \beta & \gamma \end{vmatrix} x^2 + \begin{vmatrix} . & a & c \\ a & b & d \\ \alpha & \beta & \delta \end{vmatrix} x + \begin{vmatrix} . & a & d \\ a & b & . \\ \alpha & \beta & \epsilon \end{vmatrix} = 0. \quad (\text{LIV. } 2)$$

This is what Sylvester's third rule would give. His second rule is simply a case of the third, viz., where $r = 1$; and his first rule is another case, viz., where $r = 0$. Had he followed the order of his former paper, he would have called the third rule his "leading theorem," and given the others as corollaries from it.

RICHELOT (May 1840).

[Nota ad theoriam eliminationis pertinens. *Crelle's Journal*, xxi. pp. 226–234.]

Just as Jacobi (1835) brought determinants to bear on Bezout's abridged method of eliminating x from two equations of the n^{th} degree, so did his fellow-professor Richelot, in treating of the other method of elimination, Euler's and Bezout's, discovered in the same year (1764). Euler's method, it will be remembered, consists in transforming the problem into the simpler one of eliminating a set of unknowns from a sufficient number of *linear* equations; and Richelot in a few lines (p. 227) points out that this may, of course, be done by equating to zero the determinant of the system of equations. An investigation connected therewith occupies the main portion of the paper.

Sylvester's method (1840) is described in passing, and the principle at the basis of it given. We have just seen that, when originally made known by the author, it was merely in the form of

a rule without any explanation. Although no doubt exists as to the mode in which it was obtained, still this first published description of the mode by Richelot deserves to be put on record. The whole passage in regard to it is as follows (p. 226):—

"Quam æquationem * inveniendi methodi diversæ a geometris adhibentur, ex quarum numero eius, quæ a clarissimo *Sylvester* in diario *The London and Edinburgh Philosophical Magazine and Journal of Science* nuper exposita est, mentionem faciendi hanc occasionem haud prætermittere velim. Ibi illius eliminationis problema reducitur ad problema eliminationis $m+n-1$ quantitatum ex systemate $m+n$ æquationum linearium. Multiplicata enim æquatione $f_1=0$ ex ordine per $y^{n-1}, y^{n-2}, \ldots, y^0$, nec non æquatione $f_2=0$ ex ordine per $y^{m-1}, y^{m-2}, \ldots, y^0$, adipiscimur systema $m+n$ æquationum linearium inter quantitates $y^{m+n-1}, y^{m+n-2}, \ldots, y^0$, quarum $m+n-1$ prioribus eliminatis, æquatio inter coefficientes † a' et a'' prodit. Quæ eliminatio facillime ita instituitur, ut determinantem harum $m+n$ æquationum linearium ponamus $=0$. Determinans vero, cum quantitates a' et a'' in æquationibus ipsæ tantum lineariter involvantur, et quantitates a' in n, nec non quantitates a'' in m ceteris æquationibus solis reperiantur, respectu illarum dimensiones $n$tæ est, respectuque harum $m$tæ. Unde concluditur, eam positam $=0$, esse quæsitam illam æquationem finalem $X=0$, quæ omni factore superflua careat. Notissima enim est proprietas ab *Eulero* inventa æquationis $X=0$, quod eius dimensio respectu quantitatum a' est $=n$, atque respectu quantitatum a'', $=m$, ita ut quæque functio integra evanescens, inter quantitates a' et a'', has dimensiones quadrans, pro genuina æquatione finali habenda sit." (LIV. 3)

Taking Sylvester's example,

$$\left.\begin{aligned} ax^2 + bx + c &= 0 \\ \alpha x^2 + \beta x + \gamma &= 0 \end{aligned}\right\},$$

* *I.e.*, æquationem finalem.

† The equations are taken in the form

$$f_1 = a'_m y^m + a'_{m-1} y^{m-1} + \ldots + a'_0 = 0,$$
$$f_2 = a''_n y^n + a''_{n-1} y^{n-1} + \ldots + a''_0 = 0.$$

and doing as Richelot here directs, we should first multiply both members of the first equation by x_{2-1} and by x^{1-1}, then both members of the second by x^{2-1} and by x^{1-1}, thus obtaining

$$\begin{aligned} ax^3 + bx^2 + cx \phantom{{}+c} &= 0\,, \\ ax^2 + bx + c &= 0\,, \\ \alpha x^3 + \beta x^2 + \gamma x \phantom{{}+\gamma} &= 0\,, \\ \alpha x^2 + \beta x + \gamma &= 0\,, \end{aligned}$$

and finally eliminate from these four equations x^3, x^2, x^1, by equating to zero the determinant of the system.

The statement "Ibi illius . . . . . linearium," which seems to contradict what we have above said in regard to the absence of explanation in Sylvester's paper, is not literally true. Richelot may have meant by it that Sylvester's result *implied* that the problem had been transformed as stated.

CAUCHY (1840).

[Mémoire sur l'élimination d'une variable entre deux équations algébriques. *Exercices d'analyse et de phys. math.*, i. pp. 385–422.]

After the appearance of the special papers on this subject by Jacobi, Sylvester, and Richelot, a review of the whole matter could not but be a desideratum. This was supplied by Cauchy in the singularly clear and able memoir which we have now reached. After an introduction of four pages there is an account (1) of Newton's method as expounded by Euler in 1748; (2) of Euler and Bezout's method of 1764; (3) of Bezout's abridged method; and (4) of a method* by means of the differences of the roots of the equations.

Euler and Bezout's method is shown to lead to the same determinant as Sylvester's, and the cause is made apparent. Cauchy's says (p. 389):—

"Supposons, pour fixer les idées, que les fonctions $f(x)$, $F(x)$

* Euler's, although not called so.

soient l'une du troisième degré, l'autre du second, en sorte qu'on ait

$$f(x) = ax^3 + bx^2 + cx + d\,,$$
$$F(x) = Ax^2 + Bx + C\,.$$

Alors u, v devront être de la forme

$$u = Px + Q\,,$$
$$v = px^2 + qx + r\,;$$

et, si l'on élimine x entre les deux équations

$$f(x) = 0\,, \quad F(x) = 0\,,$$

l'équation résultante sera précisément celle qu'on obtiendra, lorsqu'on choisera les coefficients

$$p,\ q,\ r,\ P,\ Q$$

de manière à faire disparaître x de la formule

$$(2) \qquad uf(x) + vF(x) = 0\,,$$

par conséquent de la formule

$$(Px+Q)f(x) + (px^2+qx+r)F(x) = 0\,,$$

que l'on peut encore écrire comme il suit :

$$(3)\ Pxf(x) + Qf(x) + px^2F(x) + qxF(x) + rF(x) = 0\,.$$

Les valeurs de

$$p,\ q,\ r,\ P,\ Q$$

qui remplissent cette condition sont celles qui vérifient les équations linéaires,

$$(4)\qquad \begin{cases} aP + Ap = 0\,, \\ bP + aQ + Bp + Aq = 0\,, \\ cP + bQ + Cp + Bq + Ar = 0\,, \\ dP + cQ + Cq + Br = 0\,, \\ + dQ + Cr = 0\,. \end{cases}$$

Donc, pour obtenir la résultante cherchée, il suffira d'éliminer les coefficients

$$P,\ Q,\ p,\ q,\ r$$

entre les équations (4), ou, ce qui revient au même, d'égaler à zéro la fonction alternée formée avec les quantités que présente le tableau

$$(5) \qquad \begin{cases} a, & 0, & A, & 0, & 0, \\ b, & a, & B, & A, & 0, \\ c, & b, & C, & B, & A, \\ d, & c, & 0, & C, & B, \\ 0, & d, & 0, & 0, & C. \end{cases}$$

On arriverait encore aux mêmes conclusions en partant de la formule (3). En effet, choisir les coefficients P, Q, p, q, r, de manière à faire disparaître de cette formule les diverses puissances

$$x, \quad x^2, \quad x^3, \ldots, x^{m+n-1},$$

de la variable x, c'est éliminer ces puissances des cinq équations,

$$(6) \quad xf(x)=0, \quad f(x)=0, \quad x^2F(x)=0, \quad xF(x)=0, \quad F(x)=0,$$

ou

$$(7) \qquad \begin{cases} ax^4 + bx^3 + cx^2 + dx = 0, \\ \qquad ax^3 + bx^2 + cx + d = 0, \\ Ax^4 + Bx^3 + Cx^2 = 0, \\ \qquad Ax^3 + Bx^2 + Cx = 0, \\ \qquad\qquad Ax^2 + Bx + C = 0. \end{cases}$$

C'est donc égaler à zéro la fonction alternée formée avec les quantités que présente le tableau,

$$(8) \qquad \begin{cases} a, & b, & c, & d, & 0, \\ 0, & a, & b, & c, & d, \\ A, & B, & C, & 0, & 0, \\ 0, & A, & B, & C, & 0, \\ 0, & 0, & A, & B, & C. \end{cases}$$

Or cette fonction alternée ne différera pas de celle que nous avons déjà mentionnée, attendu que, pour passer du tableau (5) au tableau (8), il suffit de remplacer les lignes horizontales par les lignes verticales, et réciproquement." (LIV. 4)

Bezout's abridged method for the equations

$$\left.\begin{aligned} a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0 \\ b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n = 0 \end{aligned}\right\}$$

is shown to lead to the final equation

$$S = 0,$$

where S is "une fonction alternée de l'ordre n formée avec les quantités que renferme le tableau,

$$\left\{\begin{array}{lllll} A_{0,0} & A_{0,1} & \dots & A_{0,n-2} & A_{0,n-1} \\ A_{0,1} & A_{1,1} & \dots & A_{1,n-2} & A_{1,n-1} \\ \dots & \dots & \dots & \dots & \dots \\ A_{0,n-2} & A_{1,n-2} & \dots & A_{n-2,n-2} & A_{n-2,n-1} \\ A_{0,n-1} & A_{1,n-1} & \dots & A_{n-2,n-1} & A_{n-1,n-1} \end{array}\right.\text{"}$$

in which

$$\begin{aligned} A_{0,l} &= a_0 b_{l+1} - b_0 a_{l+1}, \\ A_{1,l} &= a_1 b_{l+1} - b_1 a_{l+1} + A_{0,l+1}, \\ A_{2,l} &= a_2 b_{l+1} - b_2 a_{l+1} + A_{1,l+1}. \\ &\dots\dots\dots \end{aligned}$$

In connection with this, however, no reference is made to Jacobi's paper of 1835.

The fourth method, which occupies much the largest space (pp. 397–422), is not a determinant method.

SYLVESTER (January 1841).

[Examples of the dialytic method of elimination as applied to ternary systems of equations. *Cambridge Math. Journ.*, ii. pp. 232–236.]

In returning to extend the method, here and generally afterwards called "dialytic," Sylvester takes occasion to say that "the principle of the rule will be found correctly stated by Professor Richelot of Königsberg in a late number of *Crelle's Journal.*" It may be noted, too, that he now for the first time uses the word *determinant.*

Only the first and last of the four examples need be given, as the subject strictly belongs to the application rather than the theory of determinants. Even these, however, will suffice to show the masterly grip which Sylvester had of his own method.

"To eliminate x, y, z between the three homogeneous equations

$$\begin{array}{lr} Ay^2 - 2C'xy + Bx^2 = 0 & (1), \\ Bz^2 - 2A'yz + Cy^2 = 0 & (2), \\ Cx^2 - 2B'zx + Az^2 = 0 & (3). \end{array}$$

Multiply the equations in order by $-z^2, x^2, y^2$, add together, and divide out by $2xy$; we obtain

$$C'z^2 + Cxy - A'xz - B'yz = 0 \qquad (4).$$

By similar processes we obtain

$$\begin{array}{lr} A'x^2 + Ayz - B'yx - C'zx = 0 & (5), \\ B'y^2 + Bzx - C'zy - A'xy = 0 & (6). \end{array}$$

Between these six, treated as simple equations, the six functions of x, y, z, viz., x^2, y^2, z^2, xy, xz, yz, treated as *independent* of each other, may be eliminated; the result may be seen, by mere inspection, to come out

$$ABC(ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C') = 0,$$

or rejecting the special (N.B. not *irrelevant*) factor ABC we obtain

$$ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C' = 0.\text{"} \quad (\text{LIV. } 5)$$

The example, however satisfactory as illustrating the dialytic method, cannot be passed over without a note in regard to the unaccountable blunder made in developing the determinant involved. In later notation the determinant is

$$\begin{vmatrix} . & C & B & -2A' & . & . \\ C & . & A & . & -2B' & . \\ B & A & . & . & . & -2C' \\ A' & . & . & A & -C' & -B' \\ . & B' & . & -C' & B & -A' \\ . & . & C' & -B' & -A' & C \end{vmatrix}.$$

Now neither of the factors given by Sylvester are really factors of this, the truth being that it

$$= -2(ABC + 2A'B'C' - BB'^2 - CC'^2 - AA'^2)^2.$$

The fourth example concerns the elimination of x, y, z between the three equations

$$\left.\begin{aligned} Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy &= 0 \\ Lx^2 + My^2 + Nz^2 + 2L'yz + 2M'zx + 2N'xy &= 0 \\ Px^2 + Qy^2 + Rz^2 + 2P'yz + 2Q'zx + 2R'xy &= 0 \end{aligned}\right\} .$$

Using each of the three multipliers x, y, z with each of the three equations, we obtain nine equations linear in the ten quantities,

$$x^3,\ y^3,\ z^3,\ x^2y,\ x^2z,\ y^2x,\ y^2z,\ z^2x,\ z^2y,\ xyz .$$

Another such equation is thus necessary for success. Sylvester obtains it very ingeniously by writing the given equations in the form

$$\left.\begin{aligned} (Ax+B'z+C'y)x + (By+C'x+A'z)y + (Cz+A'y+B'x)z &= 0 \\ (Lx+M'z+N'y)x + (My+N'x+L'z)y + (Nz+L'y+M'x)z &= 0 \\ (Px+Q'z+R'y)x + (Qy+R'x+P'z)y + (Rz+P'y+Q'x)z &= 0 \end{aligned}\right\} ,$$

and then eliminating x, y, z. The work is not continued further.

We may ourselves note, in conclusion, that the fourth example includes in a sense the three others, but that it does not follow therefrom that by giving the requisite special values to the coefficients in the result of the general example, we should obtain the results for the particular examples in the forms already reached. Indeed, it is on account of this apparent non-agreement that the dialytic method is valuable to the theory of determinants, some very remarkable identities being arrived at by its aid. An explanation is also thus afforded of the trouble we have taken to elucidate its history.

CRAUFURD, A. Q. G.* (February 1841).

[On a method of algebraic elimination. *Cambridge Math. Journal*, ii. pp. 276–278.]

In Craufurd we have an independent discoverer of the dialytic method. A full account of his paper is quite unnecessary: the few

* Only the initials A. Q. G. C. are appended to the article. There can be little doubt, however, that they belong to Craufurd, whose name in full appears elsewhere in the *Journal*.

lines dealing with his introductory example will suffice to establish the fact. He says:—

"Let it be required to eliminate x from the equations

$$x^2+px+q=0\,,$$
$$x^2+p'x+q'=0\,.$$

Multiply each of the proposed equations by x, and you obtain

$$x^3+px^2+qx=0\,,$$
$$x^3+p'x^2+q'x=0\,.$$

These two combined with the two given equations make a system of four equations containing three quantities to be eliminated, viz., x, x^2, x^3; and they are of the first degree with respect to each of these quantities. We may, therefore, eliminate x, x^2, x^3 by the rules for equations of the first degree. The result is . . . ."

He enunciates a general rule, and then takes up the analogous subject in Differential Equations, where successive differentiation takes the place of successive multiplication by x. In a postscript he acknowledges Sylvester's priority which the editor had pointed out to him. He knew nothing of determinants.

CAUCHY (March 8, 1841).

[Note sur la formation des fonctions alternées qui servent à résoudre le problème de l'élimination. *Comptes Rendus . . . . Paris*, xii. pp. 414–426; or *Œuvres Complètes d'Augustin Cauchy*, 1re Sér., vi. pp. 87–99.]

Recalling the fact that the final equation, resulting from the elimination of several unknowns from a set of linear equations, has for its first member "une fonction alternée," and pointing out the further fact that the same holds good in regard to the elimination of one unknown from two equations of any degree, "puisque les methodes de Bezout et d'Euler reduisent ce denier problème au premier," Cauchy affirms the importance of being able easily to write out the full expansion of such functions. There can be little

doubt, however, that it was the second fact alone,—in other words, the discoveries of Jacobi, Sylvester, and Richelot,—which influenced the veteran Cauchy to return to a subject practically untouched by him for thirty years.

The opening part of the paper is, of course, necessarily old matter. One thing to be noted is that Cauchy tacitly discards the term *determinant*, which he was the means of introducing, using uniformly the more general expression *fonction alternée* instead. Another is that he adopts the rules of signs which makes use of the number of *interchanges*. From this his own peculiar rule of signs is deduced, and made the starting point for the fresh investigation which forms the main portion of the paper. The exposition of his rule, which differs from that of 1812, is worthy of a little attention, both on its own account and because otherwise the matter following would be scarcely intelligible. In the case of any term ("terme" or "produit") of the determinant

$$\sum \pm a_{0.0}a_{1.1}a_{2.2}a_{3.3}a_{4.4}a_{5.5}a_{6.6},$$

say the term

$$a_{0.1}a_{1.0}a_{2.5}a_{3.3}a_{4.6}a_{5.4}a_{6.2},$$

there is an underlying separation of the indices 0, 1, . . , 6 into groups ("groupes"), by reason of the system of pairing; that is to say, since an index is found paired along with one index and not with another, there arises the possibility of looking upon those which happen to be paired with one another as belonging to the same family group. Thus, attending to the first a of the term, we see that 1 and 0 belong to the same group, and as on scanning the rest of the term, we find neither of them associated with any other index, we conclude that the group is *binary* ("un groupe *binaire*"). Again, we see that 2 is paired with 5, 5 with 4, 4 with 6, and 6 with 2; this gives us the quaternary group (2, 5, 4, 6). Lastly, 3 is seen to be paired with 3, and thus forms a group by itself. Now, if we wish to find how many interchanges of the second indices are necessary in order to obtain the given term

$$a_{0.1}a_{1.0}a_{2.5}a_{3.3}a_{4.6}a_{5.4}a_{6.2}$$

from the typical term

$$a_{0.0}a_{1.1}a_{2.2}a_{3.3}a_{4.4}a_{5.5}a_{6.6},$$

we may do the counting piecemeal, attending at one time to only

that part of the term which corresponds to one of the groups of indices. In the case of the group (3), the number of interchanges is 0; in the case of the binary group (0, 1) it is 1; and in the case of the quaternary group it is 3—the number of interchanges being "évidemment" one less than the number of indices in the group. If, therefore, for a given term there be in all m groups, viz. f groups of one index each, g groups of two indices each, h of three, k of four, &c., the number of necessary interchanges will be

$$
\begin{aligned}
& 0.f + 1.g + 2.h + 3.k + \ldots, \\
\text{which} \quad = \quad & f + 2.g + 3.h + 4.k + \ldots, \\
& -(f + g + h + k + \ldots), \\
= \quad & n - m;
\end{aligned}
$$

and consequently the sign of the term will be + or -1 according as $n-m$ is even or odd. (III. 28)

The first step of the new investigation is to define "termes semblables ou de même espèce." *Two terms are said to be alike or of the same species when the one may be obtained from the other by subjecting* both *sets of indices in the latter to one and the same substitution or permutation.* Thus recurring to the term above used,

$$a_{0.1}a_{1.0}a_{2.5}a_{3.3}a_{4.6}a_{5.4}a_{6.2},$$

and substituting in both of its sets of indices 6, 0, 1, 4, 3, 2, 5, instead of 0, 1, 2, 3, 4, 5, 6 respectively,—in other words, and with the notation of the memoir of 1812, performing the substitution

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 0 & 1 & 4 & 3 & 2 & 5 \end{pmatrix},$$

we obtain the like term

$$a_{6.0}a_{0.6}a_{1.2}a_{4.4}a_{3.5}a_{2.3}a_{5.1}. \qquad \text{(LV.)}$$

The groups in two like terms are evidently similar, the values of $f, g, h, \ldots$ for the one being the same as those for the other. Indeed, since it is in this matter of groups or cycles that the terms have any likeness at all, the expression "*cyclically alike*" would have been a better term for Cauchy to use.

From the definition there arises the self-evident proposition—*Terms which are cyclically alike have the same sign.* (III. 29)

Also, *the full expansion of a determinant may be represented by writing a term of each cyclical species, and prefixing to each such typical term the symbol* Σ *with its proper sign,* + *or* − . (LV. 2)

To obtain a term of any given cyclical species, that is to say, corresponding to given values of $f, g, h, \ldots$, all the preparation that is necessary is to write the indices

$$0,\ 1,\ 2,\ 3,\ \ldots,\ (n-1),$$

enclose each of the first f of them in brackets, enclose in brackets each of the next g pairs, then each of the next h triads, and so on. This gives the groups of the term, and the term itself readily follows. For example, if we desire in the case of the determinant $\Sigma \pm a_{00}a_{11}a_{22}a_{33}a_{44}a_{55}a_{66}$ a term corresponding to $f=2$, $g=1$, $h=1$* we take the indices

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\,;$$

bracket them thus

$$(0),\ (1),\ (2,\ 3),\ (4,\ 5,\ 6)\,;$$

and with the help of this, write finally

$$a_{0.0}\,a_{1.1}\,a_{2.3}\,a_{3.2}\,a_{4.5}\,a_{5.6}\,a_{6.4}\,. \qquad \text{(II. 7)}$$

The number of different cyclical species of terms in a determinant of the n^{th} order is evidently equal to the number of positive integral solutions of the equation

$$f+2g+3h+\ .\ .\ .\ +nl=n. \qquad \text{(LV. 3)}$$

Cauchy's illustration of this is clearness itself. He says (p. 419):—

"Si, pour fixer les idées, on suppose $n=5$, alors, la valeur de n pouvant être présentée sous l'une quelconque des formes,

$$\begin{array}{l} 1+1+1+1+1, \\ 1+1+1+2, \\ 1+2+2, \\ 1+1+3, \\ 2+3, \\ 1+4, \\ 5, \end{array}$$

les systèmes de valeurs de

$$f,\ g,\ h,\ k,\ l,$$

se réduiront à l'un des sept systèmes

* It would be convenient to say, *a term of the cyclical species* $2(1)+1(2)+1(3)$.

$$\begin{array}{lllll}
f=5, & g=0, & h=0, & k=0, & l=0, \\
f=3, & g=1, & h=0, & k=0, & l=0, \\
f=1, & g=2, & h=0, & k=0, & l=0, \\
f=2, & g=0, & h=1, & k=0, & l=0, \\
f=0, & g=1, & h=1, & k=0, & l=0, \\
f=1, & g=0, & h=0, & k=1, & l=0, \\
f=0, & g=0, & h=0, & k=0, & l=1;
\end{array}$$

et par suite, une fonction alternée du cinquième ordre renfermera sept espèces de termes."

The next question considered is as to the number of terms of a given cyclical species which exist in any determinant of the n^{th} order. The species being characterised by f groups of one index each, g groups of two indices each, h groups of three indices each, &c., the required number of terms is denoted by

$$N_{f,\,g,\,h,\,\ldots,\,l}\,.$$

Now all the terms of the species will certainly be got if we write in succession the various permutations of the n indices $0, 1, 2, 3, \ldots\ldots, n-1$, and then in the usual way mark off each permutation into the specified groups, viz., first f groups of one index each, then g groups of two indices each, and so on. As a rule, however, each term of the species will, in this way, be obtained more than once. For, if we examine in its grouped form the particular permutation, which was the first to give rise to a certain term, we shall find that changes are possible upon it without entailing any change in the term. For example, the set of groups

$$(0),\ (1),\ (2, 3),\ (4, 5, 6),$$

instanced above as corresponding to the term

$$a_{0,0}a_{1,1}a_{2,3}a_{3,2}a_{4,5}a_{5,6}a_{6,4}\,,$$

might be changed into

$$(1),\ (0),\ (2, 3),\ (4, 5, 6)$$

or $$(1),\ (0),\ (3, 2),\ (6, 4, 5)$$

or $$\ldots\ldots\ldots\ldots$$

which, while still corresponding to the term

$$a_{0.0}\,a_{1.1}\,a_{2.3}\,a_{3.2}\,a_{4.5}\,a_{5.6}\,a_{6.4}\,,$$

are derivable from different permutations of the seven indices 0, 1, 2, 3, 4, 5, 6. In fact, the f groups of one index each may be permuted among themselves in every possible way, so may the g binary groups, the h ternary groups, &c. Further, with like immunity to the term, each separate group may be written in as many ways as there are indices in it,—the group (4, 5, 6), for example, being safely changeable into (5, 6, 4) or (6, 4, 5). The number, therefore, of different permutations of 0, 1, 2, 3, 4, 5, 6, which will give rise to any particular term, is

$$(1.2.3\ldots f \times 1.2.3\ldots g \times 1.2.3\ldots h \times \ldots \times 1.2.3\ldots l) \times (1^f 2^g 3^h \ldots n^l),$$

or say,

$$(f\,!\,g\,!\,h\,!\ldots l\,!)(1^f 2^g 3^h \ldots n^l).$$

There thus results the equation

$$(f\,!\,g\,!\,h\,!\ldots l\,!)(1^f 2^g 3^h \ldots n^l)\,\mathrm{N}_{f,g,h,\ldots} = n\,!,$$

whence

$$\mathrm{N}_{f,g,h,\ldots,l} = \frac{n\,!}{(f\,!\,g\,!\,h\,!\ldots l\,!)(1^f 2^g 3^h \ldots n^l)}. \qquad \text{(LV. 4)}$$

Following this interesting result a few deductions and verifications are given. First of all it is pointed out that since the total number of terms of all species is n ! we must conclude that

$$n\,! = \Sigma \frac{n\,!}{(f\,!\,g\,!\,h\,!\ldots l\,!)(1^f 2^g 3^h \ldots n^l)},$$

where $f + 2g + 3h + \ldots + nl = n.$

Cauchy says (p. 423) :—

"Cette dernière formule paraît digne d'être remarquée. Si, pour fixer les idées, on prend $n = 5$ l'équation donnera

$$\begin{aligned} 1.2.3.4.5 = \mathrm{N}_{5,0,0,0,0} + \mathrm{N}_{3,1,0,0,0} + \mathrm{N}_{1,2,0,0,0} + \mathrm{N}_{2,0,1,0,0} \\ + \mathrm{N}_{0,1,1,0,0} + \mathrm{N}_{1,0,0,1,0} + \mathrm{N}_{0,0,0,0,1}, \end{aligned}$$

et par suite

$$1.2.3.4.5 = 1 + 10 + 15 + 20 + 20 + 30 + 24 = 120,$$

ce qui est exact."

Again, since the number of positive terms in a determinant is equal to the number of negative terms, and since the terms, whose number $\mathrm{N}_{f,g,h,\ldots l}$, has just been found, have all the sign-factor

$$(-1)^{n-(f+g+h+\ldots+l)},$$

we have on leaving out the common factor $(-1)^n$ the identity

$$0=\Sigma(-1)^{f+g+h+\ldots+l}\frac{n!}{(f!g!h!\ldots l!)(1^f 2^g 3^h\ldots n^l)},$$

which like its companion may be illustrated by the case of $n=5$, viz.,

$$0=1-10+15+20-20-30+24.^*$$

Lastly, attention is directed to the fact that when n is a prime, and therefore not exactly divisible by any integer less than itself, the number

$$\frac{n!}{(f!g!h!\ldots l!)(1^f 2^g 3^h\ldots n^l)}$$

must be exactly divisible by n, except in the case

$$f=n,\ g=0,\ h=0,\ \ldots,\ l=0,$$

when it has the value 1, and in the case

$$f=0,\ g=0,\ h=0,\ \ldots,\ l=1,$$

when it has the value $(n-1)!$ It, therefore, follows from either of the two preceding identities, that the sum of these two values must be divisible by n,—which is Wilson's theorem.

The remaining two pages are occupied with the expansion of a determinant of special form, viz., that afterwards known by the name *axisymmetric*.

JACOBI (1841).

[De formatione et proprietatibus Determinantium. *Crelle's Journal*, xxii. pp. 285–318.]

The value which Jacobi attached to determinants as an instrument of research has already become well known to us: we have

* In connection with this and in illustration of a previous remark regarding a mode of expressing the full expansion of a determinant, we have

$$\begin{aligned}\Sigma\pm a_{00}a_{11}a_{22}a_{33}a_{44} = \quad & a_{00}a_{11}a_{22}a_{33}a_{44} - \Sigma a_{00}a_{11}a_{22}a_{34}a_{43}\\ &+ \Sigma a_{00}a_{12}a_{21}a_{34}a_{43} + \Sigma a_{00}a_{11}a_{23}a_{34}a_{42}\\ &- \Sigma a_{01}a_{10}a_{23}a_{34}a_{42} - \Sigma a_{00}a_{12}a_{23}a_{34}a_{41}\\ &+ \Sigma a_{01}a_{12}a_{23}a_{34}a_{40}. \qquad \text{(LV. 2)}\end{aligned}$$

found him, indeed, in almost constant employment of the functions. In the memoir now reached, however, we have still stronger evidence of his interest in the subject, and of his opinion as to its importance. Knowing of no succinct and logically arranged exposition of their properties readily accessible to mathematicians, he deliberately set himself the task of preparing a memoir to supply the want. In his few words of preface he says :—

> "Sunt quidem notissimi Algorithmi, qui aequationum linearium litteralium resolutioni inserviunt. Neque tamen video eorum proprietates praecipuas, ita breviter enarratas atque in conspectum positas esse, quantum optare debemus propter earum in gravissimis quaestionibus Analyticis usum. Scilicet illae proprietates quamvis elementares non omnes ita tritae sunt, ut quas indemonstratas relinquere deceat, et valde molestum est earum demonstrationibus altiorum ratiociniorum decursum interrumpere. Cui defectui hic supplere volo quo commodius in aliis commentationibus ad hanc recurrere possim ; neutiquam vero mihi propono totam illam materiam absolvere."

While Jacobi was aware, as we have already partly seen, of the labours of Cramer, Bezout, Vandermonde, Laplace, Gauss, and Binet, his main source of inspiration is Cauchy. Of all the writers since Cauchy's time, indeed, he is the first who gives evidence of having read and mastered the famous memoir of 1812. It scarcely needs be said, however, that his own individuality and powerful grasp are manifest throughout the whole exposition.

At the outset there is a reversal of former orders of things; Cramer's rule of signs for a permutation and Cauchy's rule being led up to by a series of propositions instead of one of them being made an initial convention or definition. This implies, of course, that a new definition of a signed permutation is adopted, and that conversely this definition must have appeared as a deduced theorem in any exposition having either of these rules as its starting point.

The new definition has its source in Cauchy, and rests on the well-known agreement as to a definite mode of forming the product P of the differences of an ordered series of quantities. This being settled to be

$$\begin{array}{r} (a_1 - a_0)(a_2 - a_0)(a_3 - a_0) \;\ldots\; (a_n - a_0) \\ (a_2 - a_1)(a_3 - a_1) \;\ldots\; (a_n - a_1) \\ (a_3 - a_2) \;\ldots\; (a_n - a_2) \\ \ldots\ldots\ldots \\ (a_n - a_{n-1}) \end{array}$$

for the quantities $a_0, a_1, a_2, \ldots a_n$, while in the order here written, the definition stands as follows (pp. 285–286):—

> "Vocemus eas indicum 0, 1, . . . , n permutationes, pro quibus P valorem eundem servat, *positivas;* eas pro quibus P valorem oppositum induit, *negativas;* sive priores dicamus pertinere ad *classem positivam permutationum,* posteriores ad *classem negativam.*"

This implies of course that the original permutation 0, 1, 2, ... , n is to be considered positive; and, such being the case, there seems to be a certain appropriateness in applying the term *negative* to a permutation whose corresponding difference-product is of the opposite sign from the difference-product corresponding to 0, 1, 2, ... , n.

The propositions which lead from the definition to Cramer's rule may be enunciated as follows :—

(*a*) One permutation performed upon another gives rise to a third, and the combined effect produced by performing the second and first in succession is the same as the effect of performing the third.

(*b*) Two given permutations belong to the same class or to opposite classes according as the permutation by means of which the one is obtained from the other belongs to the positive or negative class.

(*c*) If the same permutation be performed on a number of permutations which all belong to one class, the resulting permutations will still all belong to one class, viz., the same or the opposite according as the operating permutation is positive or negative.

(*d*) The order of compounding a set of permutations is, as a rule, not immaterial.

(*e*) The permutations which arise by compounding a set of permutations in every possible order belong all to the same class. (III. 31)

(*f*) The interchange of two indices is equivalent to the performance of a negative permutation.

(*g*) The interchange of two indices causes all the positive permutations to become negative, and all the negative to become positive.

Definition.—Two permutations may be called reciprocal which being performed in succession do not alter the order existing before the operations. (XXIV. 2)

(*h*) Reciprocal permutations belong to the same class.

In the original, it must be borne in mind, these are not separated and numbered, but appear merely as consecutive sentences in a paragraph. The words "classem negativam" of the definition above given are followed in the same line by

> "Binis propositis permutationibus quibuscunque, certa exstabit permutatio, qua post alteram adhibita altera prodit. Pertinebunt duæ permutationes propositæ ad classem eundem aut ad classes oppositas, prout permutatio, qua altera ex altera obtinetur, ad classem positivam aut negativam pertinet," &c.

—that is to say, by the propositions which have been paraphrased into (*a*), (*b*), &c.

The most essential point to be considered in connection with them is the probable meaning of the expression "permutationem adhibere," or the free English translation of it, "to perform a permutation." An example will make it clear. To perform the permutation 35412 would seem to be the operation of removing the 3rd member of a series of five things to the first place, the 5th member to the second place, the 4th member to the third place, and so on. With this explanation the proposition (*a*) is self-evident, an example of it being (if we may improvise a symbolism)

$$(35412)(41352) = (32541),$$

where 35412 is the operating permutation. Cauchy's usage, it may

be remembered, was to speak of "applying a substitution to a permutation."*

Of the proposition (*b*) a proof is given, which may be paraphrased as follows:—Let the three permutations referred to change P, the original product of differences, into e_1P, e_2P, e_3P, respectively, the *e*'s of course being either $+1$ or -1. Then as the performance of the first two permutations in succession will result in the change of P into $e_1.e_2$P, we must have

$$e_1 \cdot e_2 = e_3 ,$$

so that e_1 and e_3 have the same or opposite signs according as e_2 is $+1$ or -1; and this is virtually the proposition to be proved. (III. 30).

A demonstration of (*d*) is also given. The two permutations being A and B, *l* the first index of A, and *m* the first index of B, the performance of A on B implies that the l^{th} index in B is to take the first place, and the performance of B on A that the m^{th} index of A is to take the first place. The resulting permutations will consequently not agree in the first index, unless the l^{th} index of B is the same as the m^{th} index of A, which manifestly need not be the case.†

To prove (*f*) is of course the same as to prove that the interchange of two indices *r* and *s*, *r* being the greater, alters the sign of the product of differences; and this is done by separating the product into three portions, viz., (1) the portion which contains neither a_r nor a_s; (2) the single factor which contains both, $a_r - a_s$; and (3) the product of all the factors having either one or the other for a term. It is then asserted that the interchange of *r* and *s* cannot alter the last of these, because it is symmetrical with respect to a_r and a_s; also, that no alteration is possible in the first, and consequently that the change in the second accounts for the validity of the proposition. (III. 32)

* He says, for example (*Journ. de l'Éc. Polyt.*, x. p. 10), "Si en appliquant successivement à la permutation A, les deux substitutions $\binom{A_2}{A_3}$ et $\binom{A_4}{A_5}$, on obtient pour résultat la permutation A_6; la substitution $\binom{A_1}{A_6}$ sera équivalente au produit des deux autres et j'indiquerai cette équivalence comme il suit

$$\binom{A_1}{A_6} = \binom{A_2}{A_3}\binom{A_4}{A_5}."$$

† This also is a paraphrase of Jacobi's proof.

As for the permutations which are called reciprocal they are, exactly those whose existence we have seen noted by Rothe, and called by him "verwandte Permutationen." Jacobi's definition, however, presents them in a slightly different light, the property involved in it being readily deducible from Rothe's. The latter's illustrative example was, as may be seen on looking back,

$$\left.\begin{array}{ll} 3,\ 8,\ 5,\ 10,\ 9,\ 4,\ 6,\ 1,\ 7,\ 2 & \mathrm{A} \\ 8,\ 10,\ 1,\ 6,\ 3,\ 7,\ 9,\ 2,\ 5,\ 4 & \mathrm{B} \end{array}\right\} .$$

Now the performance of either A on B or B on A* gives rise to

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,$$

the original arrangement: consequently A and B satisfy Jacobi's definition. The proposition (h) is also Rothe's.

After these propositions, as already intimated, the subject of other rules of signs is taken up, the first rule considered being Cramer's. Since in the product of differences corresponding to any permutation every factor in which an index is preceded by a smaller index would require the sign-factor -1 to be annexed to it in order that the said product might be transformed into the original product of differences, it is clear that the determination of the class to which the permutation belongs is reduced to counting the number of such inversions. But the pairs of indices in the product of differences corresponding to the given permutation are exactly the pairs of indices to be examined in applying Cramer's rule. The identity of the two rules is thus apparent. (III. 33)

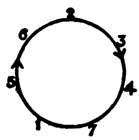
To the demonstration Jacobi adds "quam regulam olim cel. Cramer dedit ill. Laplace demonstravit." The last assertion is notable for two reasons: first, because the rule like Jacobi's own is incapable of proof being a definition, postulate, or convention according to the mode in which it is expressed: secondly, because an examination of Laplace's memoir shows that there is no ground for the statement. The fitness of the rule for the determination of the signs of the numerators and denominators of the unknowns in a set of simultaneous linear equations may of course be demonstrated, and perhaps this was in Jacobi's mind, but prior to the statement the abstract subject of permutations had alone been discussed.

* In the compounding of reciprocal permutations the order is immaterial. This is the exception hinted at in (d).

The other rule of signs dealt with is Cauchy's, in which permutation-cycles are counted instead of inversions. The existence of such cycles is the first point to be established, that is to say, it has to be shown that *any permutation* of 1 2 3 . . . *n may be obtained from any other by the performance of one or more cyclical permutations.* Let 3271654 be the permutation sought,* and 2647513 the permutation from which it is to be derived. Placing the former under the latter, thus

$$\begin{array}{ccccccc} 2 & 6 & 4 & 7 & 5 & 1 & 3 \\ 3 & 2 & 7 & 1 & 6 & 5 & 4, \end{array}$$

we see that 2 has to be changed into 3, then seeking 3 in the upper line we see that it has to be changed into 4, similarly that 4 has to be changed into 7, 7 into 1, 1 into 5, 5 into 6, and 6 into 2, the element with which we started. Now the proof turns upon the simple fact that the elements in the two lines being exactly the same, by following a string of changes like this we are bound sooner or later to reach in the second line the element we started within the first. It may be that as here one cycle



suffices for the second transformation; but if not, as in the case of the two permutations

$$\begin{array}{ccccccc} 2 & 6 & 4 & 7 & 5 & 1 & 3 \\ 4 & 1 & 5 & 7 & 2 & 3 & 6, \end{array}$$

where the short cycle 245 is obtained, we turn to the remaining elements, and knowing that those in the first line are of necessity the same as those in the second, we see that the application of the same process to them must, for the same reason as before, lead to a cycle. The possibility of arriving at any permutation by means of cyclical permutations alone is thus made manifest. The next point to be established is that *a cyclical permutation of* r *elements can be accomplished by* r − 1 *interchanges of pairs of elements.* Little more than the statement of this is necessary. For if the elements of the

* This is a paraphrase of Jacobi's demonstration, which is not so simple as it might have been. The notation of substitutions, which Jacobi did not follow Cauchy in using, is here a great help toward clearness.

cycle be $a_1, a_2, a_3, \ldots, a_r$, it is clear that to change a_1 into a_2, a_2 into a_3, &c., has the same effect as to interchange a_1 and a_2, then a_1 and a_3, then a_1 and a_4 and so on, the final interchange being that of a_1 and a_r; and there are in all $r-1$ interchanges. This being proved, the final step is taken as in Cauchy's Note of 8th March. (III. 34)

This rule of Cauchy's Jacobi deservedly characterises as beautiful. It is important, however, to take note that it possesses the other quality of usefulness in as marked a degree; and such being the case one is surprised to find that it has not received the attention which was its due. Any reader who will make a comparison of it and Cramer's by actual application of them to a number of examples will soon find that Cramer's is more lengthy and requires more care to be given to it to avoid errors.*

The preliminary subject of permutations having been thus dealt with, determinants are taken up. In the first section regarding them there is little noteworthy. Cauchy's word "terme" is supplanted by the fitter word *element*, and *term* ("terminus") is put to a more appropriate use; that is to say, $a_k^{(i)}$ is called an element of the determinant $\Sigma \pm aa'_1a''_2 \ldots a_n^{(n)}$ and $a_k a'_{k'} a''_{k''} \ldots a_{k^n}^{(n)}$ a term. Further, the word *degree* is employed in place of Cauchy's more suitable word *order*, "ipsum R dicam determinans $n+1^{\text{ti}}$ *gradus*."

A section of two pages is given to considering the effect produced upon the aggregate of terms by the vanishing of certain of the elements. The propositions enunciated, with the exception of one made use of at an earlier date by Scherk, are as follows (pp. 291, 292):—

"I. Quoties pro indicis k valoribus $0, 1, 2, \ldots, m-1$ evanescant elementa $a_k^{(m)}, a_k^{(m+1)}, \ldots \; a_k^{(n)}$, determinans

$$\Sigma \pm aa_1'a_2'' \ldots a_n^{(n)}$$

abire in productum a duobus determinantibus

$$\Sigma \pm aa_1' \ldots a_{m-1}^{(m-1)} \,.\, \Sigma \pm a_m^{(m)} a_{m+1}^{(m+1)} \ldots a_n^{(n)}. \qquad \text{(XIV. 6)}$$

* The best way perhaps of applying Cauchy's rule is to write the primitive permutation, 123456789 say, above the given permutation, 683192457 say, draw the pen through 1 and the figure below it, seek 6 in the upper line and draw the pen through it and the figure below it, and so on, marking down 1 on the completion of every cycle.

"II. Evanescentibus elementis omnibus,

$$a_k^{m}, \ a_k^{(m+1)}, \ \ldots, \ a_k^{(n)}$$

in quibus respective index inferior k indicibus superioribus $m, m+1, \ldots, n$ minor est, fieri (VI. 7)

$$\Sigma \pm aa_1'a_2'' \ldots a_n^{(n)} = a_m^{(m)}a_{m+1}^{(m+1)} \ldots a_n^{(n)} \cdot \Sigma \pm aa_1' \ldots a_{m-1}^{(m-1)}.$$

"IV. Evanescentibus elementis omnibus,

$$a_k^{(m)}, \ a_k^{(m+1)} \ \ldots \ a_k^{(n)},$$

in quibus indices inferiores superioribus minores sunt, si insuper habetur,

$$a_m^{(m)} = a_{m+1}^{(m+1)} \ \ldots \ = a_n^{(n)} = 1,$$

fit $\Sigma \pm aa_1'a_2'' \ldots a_n^{(n)} = \Sigma \pm aa_1' \ldots a_{m-1}^{(m-1)}$." (VI. 7)

As immediate deductions from the definition these are somewhat out of place, the trouble of demonstrating the first of them being virtually thrown away. The trouble taken by Jacobi, too, was less than required, the question of sign, for example, being inadequately discussed.

In the course of the next section which deals with what we have called the recurrent law of formation, and with the vanishing aggregate connected with this law, Jacobi gives an expression for the complete differential of a determinant, the elements being viewed as independent variables. The passage is (p. 293) :—

"Determinans R est singularum quantitatum $a_k^{(i)}$ respectu expressio linearis, atque ipsius $a_k^{(i)}$ coefficientem, qua in determinante R afficitur, vocavimus $A_k^{(i)}$; unde adhibita differentialium notatione ipsum $A_k^{(i)}$ exhibere licet per formulam,

$$3. \quad A_k^{(i)} = \frac{\partial R}{\partial a_k^{(i)}}.$$

Hinc si quantitatibus $a_k^{(i)}$ incrementa infinite parva tribuimus,

$$da_k^{(i)},$$

simulque R incrementum dR capit, fit

$$4. \quad dR = \Sigma A_k^{(i)} da_k^{(i)}, \qquad \text{(LVI.)}$$

siquidem sub signo summatorio utrique indici i et k valores 0, 1, 2, . . . , n conferuntur."

The recurrent law of formation and its dependent neighbour formula he is enabled, by means of (3), to view as the partial differential equations which the determinant must satisfy. His words are (p. 295):—

"Substituendo formulas (3), inventas formulas sic quoque exhibere licet:

$$9.\quad R = a^{(i)}\frac{\partial R}{\partial a^{(i)}} + a_1^{(i)}\frac{\partial R}{\partial a_1^{(i)}} + \ldots + a_n^{(i)}\frac{\partial R}{\partial a_n^{(i)}},$$

$$= a_k\frac{\partial R}{\partial a_k} + a'_k\frac{\partial R}{\partial a'_k} + \ldots + a_k^{(n)}\frac{\partial R}{\partial a_k^{(n)}},$$

$$10.\quad 0 = a^{(i')}\frac{\partial R}{\partial a^{(i)}} + a_1^{(i')}\frac{\partial R}{\partial a_1^{(i)}} + \ldots + a_n^{(i')}\frac{\partial R}{\partial a_n^{(i)}},$$

$$0 = a_{k'}\frac{\partial R}{\partial a_k} + a'_{k'}\frac{\partial R}{\partial a'_k} + \ldots + a_{k'}^{(n)}\frac{\partial R}{\partial a_k^{(n)}}.$$

Quae sunt aequationes differentiales partiales quibus determinans R satisfacit."

Passing over a section (7) on simultaneous linear equations, and a short section (8) in which Laplace's expansion-theorem is enunciated, we come to two sections dealing with what at a later time would have been called the secondary minors. No name is given to them by Jacobi; they only appear as co-factors of the product of a pair of elements, the aggregate of the terms containing $a_g^{(f)}\, a_{g'}^{(f')}$ as a factor being denoted by

$$a_g^{(f)}\, a_{g'}^{(f')}\,.\ A_{g,g'}^{f,f'}. \qquad \text{(XLI. 8)}$$

From observing that the interchange of f and f' or of g and g' alters R into $-$R and cannot alter $A_{g\ g}^{f\ f}$, it is concluded that

$$A_{g',g}^{f,f'} = A_{g,g'}^{f',f} = -A_{g,g'}^{f,f'},$$

and that the full co-factor of $A_{g,g'}^{f,f'}$ is $a_g^{(f)}\, a_{g'}^{(f')} - a_{g'}^{(f)}\, a_g^{(f')}$ in accordance with the expansion-theorem of the previous section. The

R

remark that $A_g^{(f)}$ can be expressed in terms of n of the quantities $A_{g,g'}^{f,f'}$ leads up to a curious set of equations the determinant of which belongs to the special class of determinants known afterwards as zero-axial skew determinants. The passage is (pp. 300, 301):—

"Designemus br. causa per (k, k') expressionem

$$10. \qquad A_{k,k'}^{f,f'} = (k, k'),$$

ita ut sit $$(k, k') = -(k', k).$$

Fit e (8) ipsi g substituendo numeros 0, 1, 2, . . , n:

$$11. \left\{ \begin{array}{l} A^{(f)} = \quad * \quad + a_1^{(f')}(0,1) + a_2^{(f')}(0,2) \;.\;.\; + a_n^{(f')}(0,n) \\ A_1^{(f)} = a^{(f')}(1,0) + \quad * \quad + a_2^{(f')}(1,2) \;.\;.\; + a_n^{(f')}(1,n) \\ A_2^{(f)} = a^{(f')}(2,0) + a_1^{(f')}(2,1) + \quad * \quad \;.\;.\; + a_n^{(f')}(2,n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_n^{(f)} = a^{(f')}(n,0) + a_1^{(f')}(n,1) + a_2^{(f')}(n,2) \;.\;.\; + \quad * \end{array} \right.$$

Similes formulae e (9) derivari possunt. In aequationibus (11) ipsorum $a^{(f')}$, $a_1^{(f')}$, etc. coëfficientes in diagonali positi evanescunt, bini quilibet coëfficientes diagonalis respectu symmetrice positi valoribus oppositis gaudent. Quae est species aequationum linearium memorabilis in variis quaestionibus analyticis obveniens." (LVII.)

The simple step from the expression of A_g^f as a differential coefficient to the similar expression for $A_{g,g'}^{f,f'}$ is next made (p. 301):—

"Ex ipsa enim aggregati $A_{g,g'}^{f,f'}$ definitione eruimus formulas

$$1. \quad A_{g,g'}^{f,f'} = \frac{\partial^2 R}{\partial a_g^{(f)} \partial a_{g'}^{(f')}} = -\frac{\partial^2 R}{\partial a_g^{(f')} \partial a_{g'}^{(f)}},$$

$$= \frac{\partial \,.\, A_g^{(f)}}{\partial a_{g'}^{(f')}} = \frac{\partial \,.\, A_{g'}^{(f')}}{\partial a_g^{(f)}} = -\frac{\partial \,.\, A_g^{(f')}}{\partial a_{g'}^{(f)}} = -\frac{\partial \,.\, A_{g'}^{(f)}}{\partial a_g^{(f')}}."$$

By taking the identities

$$0 = aA_k + a'A'_k + \ldots + a^{(n)}A^{(n)}_k,$$

$$0 = a_1A_k + a'_1A'_k + \ldots + a_1^{(n)}A^{(n)}_k,$$

$$\cdots\cdots\cdots\cdots$$

$$R = a_kA_k + a'_kA'_k \ \ldots + a^{(n)}_kA^{(n)}_k,$$

$$\cdots\cdots\cdots\cdots$$

$$0 = a_nA_k + a_nA'_k + \ldots + a^{(n)}_nA^{(n)}_k;$$

using the multipliers

$$A^{i,i'}_{0,k'},\ A^{i,i'}_{1,k'}, \ldots, A^{i,i'}_{k,k}, \ldots, A^{i,i'}_{n,k'},$$

and adding, there is obtained

4. $$R \,.\, A^{i,i'}_{k,k'} = A^{(i)}_k A^{(i')}_{k'} - A^{(i)}_{k'} A^{(i')}_k,$$

—a result at once recognisable as a case of the theorem regarding a minor of the adjugate. Next by starting with Bezout's identity connecting any eight quantities, the particular eight taken being

$$A^{(i)}_k,\ A^{(i)}_{k'},\ A^{(i)}_{k''},\ A^{(i)}_{k'''},$$
$$A^{(i')}_k,\ A^{(i')}_{k'},\ A^{(i')}_{k''},\ A^{(i')}_{k'''},$$

and making six substitutions of the kind

$$A^{(i)}_k A^{(i')}_{k'} - A^{(i)}_{k'} A^{(i')}_k = R \,.\, A^{i,i'}_{k,k'},$$

just seen to be valid, there arises the identity

$$A^{i,i'}_{k,k'} A^{i,i'}_{k''k'''} + A^{i,i'}_{k,k''} A^{i,i'}_{k''',k'} + A^{i,i'}_{k,k'''} A^{i,i'}_{k',k''} = 0. \qquad \text{(XXIII. 11)}$$

This clearly belongs to the class of vanishing aggregates of products of pairs of determinants; but in order that its true character may be seen, and comparison made possible between it and others of the same class already obtained, a more lengthy notation is necessary. Taking for shortness the case where the primitive determinant is of the 8th order, but writing it in the form

$$|a_1b_2c_3d_4e_5f_6g_7h_8|$$

and making

$$i, i' = 3, 6 \quad \text{and} \quad k, k', k'', k''' = 5, 6, 7, 8,$$

we find the identity to be

$$|a_1b_2d_3e_4g_7h_8|\cdot|a_1b_2d_3e_4g_5h_6| - |a_1b_2d_3e_4g_6h_8|\cdot|a_1b_2d_3e_4g_5h_7| + |a_1b_2d_3e_4g_6h_7|\cdot|a_1b_2d_3e_4g_5h_8| = 0,$$

a glance at which suffices to show that it is nothing more than the extensional of

$$|g_7h_8|\cdot|g_5h_6| - |g_6h_8|\cdot|g_5h_7| + |g_6h_7|\cdot|g_5h_8| = 0,$$

the very identity of Bezout which was taken as a basis for it. As the same extensional has already been found among those of Desnanot, any new interest in it is due to the peculiar way in which Jacobi obtained it. By the same method, viz., by substituting for secondary minors an expression (4) involving primary minors and the primitive determinant, he shows that

$$A_k^{(i)}A_{k',k''}^{i,i'} + A_{k'}^{(i)}A_{k''k,}^{i,i'} + A_{k''}^{(i)}A_{k,k'}^{i,i'} = 0. \qquad \text{(XXIII. 12)}$$

This being translated in the same manner as the preceding, becomes

$$|a_1b_2d_3e_4f_6g_7h_8|\cdot|a_1b_2d_3e_4g_5h_8| - |a_1b_2d_3e_4f_5g_7h_8|\cdot|a_1b_2d_3e_4g_6h_8| + |a_1b_2d_3e_4f_5g_6h_8|\cdot|a_1b_2d_3e_4g_7h_8| = 0,$$

and is thus seen to be another of Desnanot's results, viz., the extensional of

$$|f_6g_7|\cdot g_5 - |f_5g_7|\cdot g_6 + |f_5g_6|\cdot g_7 = 0. \qquad \text{(XXIII. 12)}$$

The deduction

$$\frac{\partial \cdot \dfrac{A_{k'}^{(i)}}{A_k^{(i)}}}{\partial\, a_{k''}^{(i')}} = -\frac{A_{k''}^{(i)}\,A_{k,k'}^{i,i'}}{A_k^{(i)}\,A_k^{(i)}}, \qquad \frac{\partial \cdot \dfrac{A_k^{(i)}}{A_k^{(i')}}}{\partial\, a_{k'}^{(i'')}} = -\frac{A_k^{(i'')}\,A_{k,k'}^{i,i'}}{A_k^{(i)}\,A_k^{(i)}},$$

is made from it by substituting appropriate differential coefficients for the primary and secondary minors involved in it. (LVIII.)

The eleventh section is devoted to the establishment of the general theorem which includes the theorem

$$R\,.\,A_{k,k'}^{i,i'} = A_k^{(i)}A_{k'}^{(i')} - A_{k'}^{(i)}A_k^{(i')}$$

of the preceding section, and which, as we have seen, Jacobi had first enunciated in 1833. To start with it is repeated that the system of equations

$$\left.\begin{array}{l}
a\ t + a_1\ t_1 + \ldots + a_k\ t_k + a_{k+1} t_{k+1} + \ldots + a_n\ t_n = u, \\
a'\ t + a_1'\ t_1 + \ldots + a'^k t_k + a_k'{}_{+1} t_{k+1} + \ldots + a_n'\ t_n = u_1, \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
a^{(k)} t + a_1^{(k)} t_1 + \ldots + a_k^{(k)} t_k + a_{k+1}^{(k)} t_{k+1} + \ldots + a_n^{(k)} t_n = u_k, \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
a^{(n)} t + a_1^{(n)} t_1 + \ldots + a_k^{(n)} t_k + a_{k+1}^{(n)} t_{k+1} + \ldots + a_n^{(n)} t_n = u_n,
\end{array}\right\}$$

gives rise to the system

$$\left.\begin{array}{l}
\mathrm{A}\ u + \mathrm{A}' u_1 + \ldots + \mathrm{A}^{(k)} u_k + \mathrm{A}^{(k+1)} u_{k+1} + \ldots + \mathrm{A}^{(n)} u_n = \mathrm{R} \,.\, t \\
\mathrm{A}_1 u + \mathrm{A}_1' u_1 + \ldots + \mathrm{A}_1^{(k)} u_k + \mathrm{A}_1^{(k+1)} u_{k+1} + \ldots + \mathrm{A}_1^{(n)} u_n = \mathrm{R} \,.\, t_1 \\
\cdot\ \cdot \\
\mathrm{A}_k u + \mathrm{A}_k' t_1 + \ldots + \mathrm{A}_k^{(k)} u_k + \mathrm{A}_k^{(k+1)} u_{k+1} + \ldots + \mathrm{A}_k^{(n)} u_n = \mathrm{R} \,.\, t_k \\
\cdot\ \cdot \\
\mathrm{A}_n u + \mathrm{A}_n' u_1 + \ldots + \mathrm{A}_n^{(k)} u_k + \mathrm{A}_n^{(k+1)} u_{k+1} + \ldots + \mathrm{A}_n^{(n)} u_n = \mathrm{R} \,.\, t_n
\end{array}\right\}$$

in which

$$\mathrm{R} = \Sigma \pm a a'_1 \ldots . a_n^{(n)}, \qquad \mathrm{A}_n^{(n)} = \Sigma \pm a a'_1 \ldots . a_{(n-1)}^{(n-1)}.$$

Then taking only the first $k+1$ equations of the first system and eliminating $t, t_1, \ldots, t_{k-1}$, there is obtained

$$\mathrm{C}_k t_k + \mathrm{C}_{k+1} t_{k+1} + \ldots . + \mathrm{C}_n t_n = \mathrm{D}u + \mathrm{D}_1 u_1 + \ldots + \mathrm{D}_k u_k, \qquad \text{(X)}$$

where the multipliers $\mathrm{D}, \mathrm{D}_1, \ldots, \mathrm{D}_k$, by which the elimination is effected, are

$$(-1)^k \Sigma \pm a' a_1'' \ldots a_{k-1}^k,$$
$$(-1)^{k+1} \Sigma \pm a a_1' \ldots a_{k-1}^k,$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$+\Sigma \pm a a'_1 a''_2 \ldots a_{k-1}^{(k-1)},$$

and consequently by $\mathrm{C}_k, \mathrm{C}_{k+1}, \ldots, \mathrm{C}_n$ are denoted

$$\Sigma \pm a a'_1 a''_2 \ldots a_{k-1}^{(k-1)} a_k^{(k)},$$
$$\Sigma \pm a a'_1 a''_2 \ldots a_{k-1}^{(k-1)} a_{k+1}^{(k)},$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$\Sigma \pm a a'_1 a''_2 \ldots a_{k-1}^{(k-1)} a_n^{(k)}.$$

Similarly, taking only the last $n-k+1$ equations of the second system and eliminating $u_{k+1}, u_{k+2}, \ldots, u_n$ there is obtained

$$\mathrm{E}u + \mathrm{E}_1 u_1 + \ldots + \mathrm{E}_k u_k = \mathrm{RF}_k t_k + \mathrm{RF}_{k+1} t_{k+1} + \ldots + \mathrm{RF}_n t_n, \qquad \text{(Y)}$$

where the multipliers $\mathrm{F}_k, \mathrm{F}_{k+1}, \ldots, \mathrm{F}_n$ by which the elimination is effected are

$$\Sigma \pm \mathrm{A}_{k+1}^{(k+1)} \mathrm{A}_{k+2}^{(k+2)} \ldots \mathrm{A}_n^{(n)},$$
$$-\Sigma \pm \mathrm{A}_k^{(k+1)} \mathrm{A}_{(k+2)}^{(k+2)} \ldots \mathrm{A}_n^{(n)},$$
$$\cdots\cdots\cdots\cdots$$
$$(-1)^{n-k} \Sigma \pm \mathrm{A}_k^{(k+1)} \mathrm{A}_{k+1}^{(k+2)} \ldots \mathrm{A}_{n-1}^{(n)},$$

and consequently by $\mathrm{E}, \mathrm{E}_1, \ldots, \mathrm{E}_k$ are denoted

$$\Sigma \pm \mathrm{A}_k \mathrm{A}_{k+1}^{(k+1)} \ldots \mathrm{A}_n^{(n)},$$
$$\Sigma \pm \mathrm{A}_k' \mathrm{A}_{k+1}^{(k+1)} \ldots \mathrm{A}_n^{(n)},$$
$$\cdots\cdots\cdots$$
$$\Sigma \pm \mathrm{A}_k^{(k)} \mathrm{A}_{k+1}^{(k+1)} \ldots \mathrm{A}_n^{(n)}.$$

These two derived equations (X), (Y), however, must be identical, because they may be both viewed as giving t_k in terms of $t_{k+1}, t_{k+2}, \ldots, t_n, u, u_1, \ldots, u_k$, and, as the first system of equations shows, this can only be done in one way. We thus have the deduction

$$\frac{\mathrm{D}_k}{\mathrm{C}_k} = \frac{\mathrm{E}_k}{\mathrm{R}.\mathrm{F}_k},$$

$$i.e. \quad \frac{\Sigma \pm a a_1' a_2'' \ldots a_{k-1}^{(k-1)}}{\Sigma \pm a a_1' a_2'' \ldots a_k^{(k)}} = \frac{\Sigma \pm \mathrm{A}_k^{(k)} \mathrm{A}_{k+1}^{(k+1)} \ldots \mathrm{A}_n^{(n)}}{\mathrm{R}\Sigma \pm \mathrm{A}_{k+1}^{(k+1)} \mathrm{A}_{k+2}^{(k+2)} \ldots \mathrm{A}_n^{(n)}}.$$

This is the keystone of the demonstration. The simple continuation of it may for sake of historical colour be given in Jacobi's own words (p. 304): *—

"In hac formula generali ipsi k tribuendo valores

$$n-1,\; n-2,\; n-3,\; \ldots,\; 1,$$

prodit:

* The demonstration in the original is considerably disfigured by misprints.

$$\left\{\begin{aligned}
&\frac{\Sigma \pm aa_1' \ldots a_{n-2}^{(n-2)}}{\Sigma \pm aa_1' \ldots a_{n-1}^{(n-1)}} = \frac{\Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)}}{RA_n^{(n)}},\\
&\frac{\Sigma \pm aa_1' \ldots a_{n-3}^{(n-3)}}{\Sigma \pm aa_1' \ldots a_{n-2}^{(n-2)}} = \frac{\Sigma \pm A_{n-2}^{(n-2)} A_{n-1}^{(n-1)} A_n^{(n)}}{R\Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\qquad\frac{a}{\Sigma \pm aa_1'} = \frac{\Sigma \pm A_1' A_2'' \ldots A_n^{(n)}}{R\Sigma \pm A_2'' A_3''' \ldots A_n^{(n)}}.
\end{aligned}\right.$$

"Harum aequationum prima suppeditat,

$$\Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)} = R\Sigma \pm aa_1' \ldots a_{n-2}^{(n-2)} = RA_{n-1,\,n}^{n-1,\,n},$$

quae cum formula (4) § pr. convenit. Deinde aequationum (10) duas, tres, quatuor etc. primas inter se multiplicando, prodit formularum systema hoc:

$$\left\{\begin{aligned}
\Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)} &= R\,\Sigma \pm aa_1' \ldots a_{n-2}^{(n-2)},\\
\Sigma \pm A_{n-2}^{(n-2)} A_{n-1}^{(n-1)} A_n^{(n)} &= R^2\Sigma \pm aa_1' \ldots a_{n-3}^{(n-3)},\\
&\ldots\ldots\ldots\ldots\\
\Sigma \pm A_1' A_2'' \ldots A_n^{(n)} &= aR^{n-1}.
\end{aligned}\right.$$

"Quas formulas amplectitur formula generalis,

$$\Sigma \pm A_{k+1}^{(k+1)} A_{k+2}^{(k+2)} \ldots A_n^{(n)} = R^{n-k-1}\Sigma \pm aa_1' \ldots a_k^{(k)}." \qquad \text{(xx. 5)}$$

Cauchy's theorem

$$\Sigma \pm AA_1' \ldots A_n^{(n)} = R^n,$$

which may be viewed as the ultimate case of this, Jacobi arrives at by expressing $\Sigma \pm A A_1' \ldots A_n^{(n)}$ in terms of $A, A_1, \ldots, A_n$ and their cofactors, substituting for the said cofactors their equivalents as just obtained, viz.

$$aR^{n-1},\quad a_1R^{n-1},\quad a_2R^{n-1},\quad \ldots\ldots,\quad a_nR^{n-1},$$

and then using the identity

$$Aa + A_1a_1 + \ldots + A_na_n = R.$$

Passing over the twelfth section, which relates to certain special

systems of equations, we come to two sections devoted to the multiplication-theorem. Of the five formally enunciated propositions which they contain, two, the second and fourth, need not be more than referred to, as their substance comes from Binet and Cauchy, and as the mode in which they are established will be sufficiently understood from the treatment of one of the others. The general problem of the two sections is the investigation of the determinant

$$\Sigma \pm c c_1 c_2'' \ldots . c_n^{(n)},$$

where

$$c_k^{(i)} = a^{(i)} a^{(k)} + a_1^{(i)} a_1^{(k)} + \ldots . + a_p^{(i)} a_p^{(k)}.$$

Taking a single term of the determinant, we have of course

$$\begin{aligned} c c_1' c_2'' \ldots c_n^{(n)} = \ & (a a + a_1 a_1 + \ldots . + a_p a_p) \\ & \times (a' a' + a_1' a_1' + \ldots . + a_p' a_p') \\ & \ldots \ldots \ldots \ldots \ldots \\ & \times (a^{(n)} a^{(n)} + a_1^{(n)} a_1^{(n)} + \ldots . + a_p^{(n)} a_p^{(n)}), \end{aligned}$$

and we see that if the multiplications indicated on the right be performed there must arise a series of $(p+1)^{n+1}$ terms of the type

$$a_r a_r \,.\, a_s' a_s' \,.\, a_t'' a_t'' \ldots \ldots \, a_w^{(n)} a_w^{(n)},$$

or by alteration of the order of the factors

$$a_r a_s' a_t'' \ldots a_w^{(n)} \,.\, a_r a_s' a_t'' \ldots a_w^{(n)},$$

where each of the inferior indices $r, s, t, \ldots, w$ may be any member of the series $0, 1, 2, \ldots, p$. If we bear in mind the meaning which we thereby assign to the summatory symbol S we may write this in the form

$$c c_1' c_2'' \ldots . c_n^{(n)} = \mathrm{S}(a_r a_s' a_t'' \ldots a_w^{(n)} \,.\, a_r a_s' a_t'' \ldots a_w^{(n)}).$$

The next point to consider is the transition from the single term $c c_1' c_2'' \ldots c_n^{(n)}$ to the full aggregate $\Sigma \pm c c_1' c_2'' \ldots c_n^{(n)}$. A glance at the sum of terms denoted by $c_k^{(i)}$ shows that by permuting the superior indices of $c c_1' c_2'' \ldots c_n^{(n)}$, the superior indices of the a's are subjected to the same permutation, and that, on the other hand, when we permute the inferior indices of $c c_1' c_2'' \ldots c_n^{(n)}$ it is the a's

that are affected, the like permutation being given to the superior indices. Making the choice of the *superior* indices of the c's, let us permute them in every possible way, and to each term thus derived from $cc_1'c_2'' \ldots c_n^{(n)}$ prefix the sign + or − according as its superior indices constitute a positive or negative permutation. By so doing the left-hand side of our identity becomes $\Sigma \pm cc_1'c_2'' \ldots c_n^{(n)}$; and, owing to the consequent permutation of the superior indices of the a's, each term on the right-hand side gives rise to $1.2.3 \ldots (n+1)$ terms whose signs are the same as the signs of the terms corresponding to them on the left hand side;—in other words, each term $a_r a_s' a_t'' \ldots a_w^{(n)} . a_r a_s' a_t'' \ldots a_w^{(n)}$ gives rise to the compound term

$$a_r a_s' a_t'' \ldots a_w^{(n)} . \Sigma \pm a_r a_s' a_t'' \ldots . a_w^{(n)} .$$

We thus reach the result

$$\Sigma \pm cc_1'c_2'' \ldots c_n^{(n)} = \mathrm{S}\left(a_r a_s' a_t'' \ldots a_n^{(n)} . \Sigma \pm a_r a_s' a_t'' \ldots a_w^{(n)}\right).$$

Although the number of terms on the right is the same as before, viz. $(p+1)^{n+1}$, arising from giving to each of the $n+1$ indices $r, s, t, \ldots, w$ any one of the $p+1$ values $0, 1, 2, \ldots, p$, it has now to be noticed that a goodly proportion of them must vanish because of the fact that $\Sigma \pm a_r a_s' a_t'' \ldots a_n^{(n)} = 0$ when any two of its inferior indices are alike. The right-hand side will thus not be altered in substance if the summatory symbol be now taken to mean that $r, s, t, \ldots, w$ are to be any $n+1$ of the $p+1$ indices $0, 1, 2, \ldots, p$. If p be less than n it will be impossible to have $r, s, t, \ldots, w$ all different, so that in that case the right-hand side must be 0. This is Jacobi's first proposition, and it constitutes his addition to the multiplication-theorem. His formal enunciation of it is (p. 309):—

"Sit

$$c_k^{(i)} = a^{(i)} a^{(k)} + a_1^{(i)} a_1^{(k)} + \ldots + a_p^{(i)} a_p^{(k)},$$

quoties $p < n$ evanescit determinans

$$\Sigma \pm cc_1'c_2'' \ldots . c_n^{(n)}.\text{"} \qquad \text{(XVIII. 6)}$$

The consideration of the case when $p = n$ leads to his second pro-

position. The natural addendum is then made regarding the multiplication of more than two determinants of the same degree (p. 310):—

> "Datis quotcunque eiusdem gradus determinantibus, eorum productum ut eiusdem gradus exhiberi posse determinans, cuius elementa expressiones sint rationales integrae elementorum determinantium propositorum." (XVII. 7)

The equally natural transition to the subject of the multiplication of two determinants of different degrees results in the proposition (p. 311):—

> "Sit pro indicis i valoribus 0, 1, 2, ... , m,
>
> $$c^{(i)}_k = a^{(i)}a^{(k)} + a^{(i)}_1 a^{(k)}_1 + \ldots + a^{(i)}_n a^{(k)}_n,$$
>
> pro indicis i valoribus maioribus quam m,
>
> $$c^{(i)}_k = a^{(k)}_i + a^{(i)}_{i+1} a^{(k)}_{i+1} + a^{(i)}_{i+2} a^{(k)}_{i+2} + \ldots + a^{(i)}_n a^{(k)}_n:$$
>
> erit

$$\Sigma \pm aa_1' \ldots a^{(m)}_m . \Sigma \pm aa_1' \ldots a^{(n)}_n = \Sigma \pm cc_1'c_2'' \ldots c^{(n)}_n.\text{''} \quad (\text{XVII. } 8)$$

Proposition IV. concerns the case where $p > n$. Proposition V. is but a corollary to the combined propositions I., II., IV., its subject being the effect of the specialisation

$$a^{(i)}_k = a^{(i)}_k.$$

The enunciation is as follows (p. 312):—

> "Posito
>
> $$c^{(i)}_k = c^{(k)}_i = a^{(i)}a^{(k)} + a^{(i)}_1 a^{(k)}_1 + \ldots + a^{(i)}_p a^{(k)}_p,$$
>
> sit determinans
>
> $$\Sigma \pm cc_1' \ldots c^{(n)}_n = \mathrm{P};$$
>
> ubi $p < n$ fit
>
> $$\mathrm{P} = 0;$$
>
> ubi $p = n$ fit
>
> $$\mathrm{P} = \left\{\Sigma \pm aa_1' \ldots a^{(n)}_n\right\}^2;$$
>
> ubi $p > n$ fit
>
> $$\mathrm{P} = \mathrm{S}\left\{\Sigma \pm a_m a'_{m'} \ldots a^{(n)}_{m^{(n)}}\right\}^2,$$
>
> siquidem pro indicibus inferioribus m, m' &c. sumuntur quilibet $n+1$ diversi e numeris 0, 1, 2 . . . . p." (XVIII. 7)

The two remaining sections (15 and 16) deal with a special system of simultaneous linear equations, interesting application being made to the theory of the Method of Least Squares.

It is important to note, in conclusion, that from one point of view Jacobi's memoir was but the introduction to two others of really greater importance, both treating of a special class of determinants. The first concerns determinants of the kind afterwards deservedly associated with his name, and bears the title "*De determinantibus functionalibus.*" It occupies the forty-one pages (pp. 319–359) immediately following the general memoir. The other, with the title "*De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum,*" treats of those determinants, first considered by Cauchy, in which the members of one set of indices represent powers, and to which the name *alternants* afterwards came to be assigned. It extends to twelve pages (pp. 360–371). The three memoirs together constitute an excellent treatise on the subject, and are known to have been markedly influential in spreading a knowledge of it among mathematicians.

CAUCHY (1841).

[Note sur les diverses suites que l'on peut former avec des termes donnés. *Exercices d'analyse et de phys. math.*, ii. pp. 145–150.]

[Mémoire sur les fonctions alternées et sur les sommes alternées. *Exercices d'analyse et de phys. math.*, ii. pp. 151–159.]

[Mémoire sur les sommes alternées, connues sous le nom de résultantes. *Exercices d'analyse et de phys. math.*, ii. pp. 160–176.]

[Mémoire sur les fonctions différentielles alternées. *Exercices d'analyse et de phys. math.*, ii. pp. 176–187.]

From internal evidence there can be little doubt that this series of papers, containing the fundamental conceptions and salient propositions of the theory of determinants, was prompted by the appearance of Jacobi's memoirs, and by the consequent conviction that the work of 1812 had begun to bear fruit. The first paper, called a "note," is introductory, on the subject of signed permutations; the three others, called "memoirs," correspond to Jacobi's,—

the first of them to Jacobi's third, the second to Jacobi's first, and the third to Jacobi's second.

The note, although on so trite a subject as the division of permutations into positive and negative, is most interesting. Cauchy's original stand-point with regard to the subject is so far unaltered that the rule of signs specially known by his name is made fundamental, and all others deduced from it. The explanations preparatory for the rule are, however, on the lines of his paper of 1840, that is to say, it is *groups* and not *circular substitutions* that are spoken of. The preference is a little difficult to justify; for notwithstanding Cauchy's assertion that groups come naturally into evidence, the idea is far-fetched as compared with that of circular substitutions. He says (p. 145):—

> "Si l'on compare une quelconque des nouvelles suites* à la première, on se trouvera naturellement conduit par cette comparaison à distribuer les divers termes
>
> $$a,\ b,\ c,\ d\ \ldots.$$
>
> en plusieurs groupes, en faisant entrer deux termes dans un même groupe, toutes les fois qu'ils occuperont le même rang dans la première suite et dans la nouvelle, et en formant un groupe isolé de chaque terme qui n'aura pas changé de rang dans le passage d'une suite à l'autre."

The question of the natural order of ideas and the best mode of presentment is really, however, of small importance, for in application a *group* and a *circular substitution* are essentially the same. The difference is entirely one of stand-point, nomenclature, and notation. The permutation

$$e,\ a,\ b,\ d,\ c,\ g,\ f,$$

being in question, and comparison between it and the primitive permutation,

$$a,\ b,\ c,\ d,\ e,\ f,\ g,$$

having been instituted, we are directed to form the members ("termes") of the permutation into groups, commencing to form a group with e and a, because they occupy like positions in the two

* *I.e.*, permutations of $a,b,c,d,\ldots$

permutations, putting b in the same group because it occupies the same position in the second permutation as one already in the group occupies in the first permutation, putting c in for the same reason, making d constitute a group by itself, and finally putting f and g together to form a third group. We are directed further, to write the members of each group in such an order that any member and the one following it may be found to occupy like positions in the primitive and derived permutations respectively. The result thus is

$$(a, e, c, b), \quad (d), \quad (f, g),$$

or $$(e, c, b, a), \quad (d), \quad (g, f),$$

or $$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .,$$

it being possible to write the first group in four ways, and the last in two. Now all this is nothing more than an unreasoning way of arriving at the circular substitutions which are necessary for the derivation of the given permutation from the primitive one. Cauchy himself, indeed, in pointing out that there would only be one way of writing a group if the members were disposed in a circumference instead of in a straight line, says:—"C'est par ce motif que dans le tome x du *Journal de l'École Polytechnique* j'ai désigné sous le nom de *substitution circulaire* l'opération qui embrasse le système entier des remplacements indiqués par un même groupe." It must be borne in mind, however, that not only the operation, but the symbol of the operation, was so denoted, and such being the case, we may then very pertinently ask, What is a group in Cauchy's usage but the symbol of a circular substitution?

The peculiarity of using the number of groups to separate the various permutations of $a, b, c, d, \ldots .$ into two classes makes its appearance in the following sentence (p. 147):—

> "De plus, ces mêmes suites ou arrangements se partageront en deux classes bien distinctes, la comparaison de chaque nouvel arrangement au premier
>
> $$a, b, c, d, \ldots .$$
>
> pouvant donner naissance à un nombre pair ou à un nombre impair de groupes."

Of course, the primitive permutation is looked upon as having its groups also, viz., one for every letter in the permutation.

Then comes the important proposition—*The interchange of two letters increases or diminishes the number of groups (substitution-cycles) by unity.* In proving it the two letters are first taken in different groups,

$$(a,b,c, \ldots, h,k), \quad (l,m,n, \ldots, r,s);$$

and since any member of a group may occupy the first place, the letters a and l are fixed upon. Now what the groups imply is that the letters

$$a, b, c, \ldots . h, k, l, m, n, \ldots . r, s$$

in the primitive permutation are changed into

$$b, c, \quad \ldots . k, a, m, n, \quad \ldots . s, l$$

respectively to form the given permutation. If therefore in the given permutation the letters a and l be interchanged, the new permutation so obtained will be got from the primitive by changing

$$a, b, c, \ldots, h, k, l, m, n, \ldots, r, s$$

into

$$b, c, \ldots . . k, l, m, n, \ldots . . . , s, a;$$

that is to say, by the changes indicated by the single group

$$(a,b,c, \ldots, h,k,l,m,n, \ldots, r,s).$$

The interchange of two letters belonging to different groups is thus seen to reduce the number of groups by one. On the other hand, it is clear that had this single group belonged to the given permutation, the interchange of two letters, a and l say, would have had the effect of breaking up the group into two,

$$(a,b,c, \ldots, h,k) \text{ and } (l,m,n, \ldots, r,s).$$

The theorem is thus established. (III. 35)

It is next pointed out that the transformation of the primitive permutation into any other may be accomplished by interchanges only, because by this means any given letter may be made to occupy the first place, then any other given letter to occupy the second place, and so on. From this also it follows that any system of circular substitutions may be replaced by a system of interchanges. Should the transformation of one permutation into

another be effected by interchanges, the number of these will be even or odd according as the two permutations belong to the same or different classes; for, by the above theorem, every interchange makes only one group more or one group less, and consequently the total number of interchanges, and the net increase or diminution of the number of groups, must be both even or both odd. The counting of *interchanges* may thus be substituted for the counting of cycles. (III. 36)

Finally, Cramer's rule is introduced, in which, as we know, it is neither cycles nor interchanges that are counted, but *inverted pairs*, or, as Cauchy, like Gergonne, calls them, *inversions*. To establish the rule, it is clear that two courses were open, viz., to connect inversions directly with cycles or to connect them with interchanges. The latter course is taken, the requisite connecting theorem being that *the interchange of two elements of a permutation increases or diminishes the number of inversions by an odd number*, an odd number of interchanges thus corresponding to an odd number of inversions, and an even to an even. The proof is not direct, like Rothe's, being effected with the help of a fourth related entity, the difference-product. The order of thought in it is as follows:—If we define the difference-product of the primitive permutation $a, b, c, d, \ldots$ to be

$$(a-b)(a-c)\ldots.(b-c)\ldots.,$$

then it is clear that in the difference-product of any derived permutation there will be found exactly as many factors with changed sign as there are inversions of order in the permutation. A change of sign in the difference-product thus becomes a test for the existence of an odd number of inversions, and consequently, instead of the theorem just enunciated, it will suffice to show that *the interchange of two elements of a permutation alters the sign of the difference-product.* This Cauchy says must be true, for, the elements being h and k, it is manifest that the factor which involves them both,

$$h-k \quad \text{or} \quad k-h,$$

must change sign, but that the factors which involve them and any third element s constitute a partial product

$$(h-s)(k-s) \quad \text{or} \quad (h-s)(s-k),$$

the sign of which cannot change. (III. 37)

Of the three memoirs, the first and third, like Jacobi's third and second, do not at present require attention. A slight reference to the first—on alternating functions—is, however, necessary, because Cauchy, unlike Jacobi, makes determinants a special class of alternating functions, and it is therefore of importance to see the exact position he assigns to them. It will be remembered that in 1812 he partitioned symmetric functions into permanent and alternating, and made determinants a class of the latter; that is to say, his scheme of logical relationship was

$$\text{Functions}\left\{\begin{array}{l}\text{(A) Symmetric}\left\{\begin{array}{l}(a)\ \text{Alternating}\left\{\begin{array}{l}(\alpha)\ \text{Determinants.}\\ \\ \end{array}\right.\\ (b)\ \text{Permanent}\end{array}\right.\\ \text{(B)}\end{array}\right.$$

The memoirs we have now come to indicate a departure from this, both verbal and substantial. The change is made too without any reason being assigned; indeed, there is not even a word to imply that any change had taken place. Alternating functions are, as in his *Cours d'analyse*, put on the same level as symmetric functions; the term *permanent* is dispensed with; a new entity, *alternating aggregates*, is introduced; what were formerly called determinants are made a class of these alternating aggregates; and for the name determinant *resultant* is substituted. The scheme of relationship is thus transformed into

$$\text{Functions}\left\{\begin{array}{l}\text{(A) Alternating}\left\{\begin{array}{l}(a)\ \text{Alternating Aggregates}\left\{\begin{array}{l}(\alpha)\ \text{Resultants.}\\ (\beta)\end{array}\right.\\ (b)\end{array}\right.\\ \text{(B) Symmetric}\\ \text{(C)}\end{array}\right.$$

Neither scheme, we must at the same time remember, is really as simple as here indicated, being complicated by the fact that a function may be alternating in more than one way. This is brought out much more explicitly and clearly in the present memoirs than in that of 1812, as the following quotations will show. We have first of all (p. 151), an *alternating function of several variables.*

> "Une fonction alternée de plusieurs variables x, y, z, . . . , est celle qui change de signe, en conservant, au signe près, la même valeur lorsqu'on échange deux de ces variables entre elles."

Next we have an *alternating function with respect to several indices* (p. 155):—

"Quelquefois on représente ces mêmes variables par une seule lettre affectée de divers indices

$$0,\ 1,\ 2,\ 3,\ \ldots\ldots,\ n,$$

et l'on peut dire alors que la fonction ou la somme dont il s'agit est *alternée par rapport à ces indices.* Ainsi, par exemple, le produit

$$(x_0 - x_1)(x_0 - x_2)(x_1 - x_2)$$

est une fonction alternée par rapport aux variables

$$x_0,\ x_1,\ x_2,$$

ou, ce qui revient au même, par rapport aux indices

$$0,\ 1,\ 2."$$

This example being an alternating function according to the first definition, it would seem that here we have a mere abbreviation or variation of language. There are, however, it must be borne in mind, functions which are alternating with respect to indices, and are not alternating according to the first definition. For example, any determinant, like

$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2,$$

is alternating with respect to all the indices involved, but is not alternating with respect to all or any other number of the variables $a_1,a_2,a_3,\ b_1,b_2,b_3,\ c_1,c_2,c_3$. Strange to say, Cauchy makes no mention of this, but goes on to a third definition, by means of which alternating functions are made in another way to include determinants. He says (p. 156):—

"On pourrait obtenir aussi des fonctions qui seraient *alternées par rapport à diverses suites*, c'est à dire, des fonctions qui auraient la propriété de changer de signe, en conservant, au signe près, la même valeur quand on échangerait entre eux les termes correspondants de ces mêmes suites. Considérons, par exemple, m suites différentes composées chacune de n termes qui se trouvent représentés, pour la première suite, par

$$x_0,\ x_1,\ \ldots\ldots,\ x_{n-1},$$

pour la seconde suite, par

$$y_0,\ y_1, \ldots\ldots, y_{n-1},$$

pour la troisième suite, par

$$z_0,\ z_1, \ldots\ldots, z_{n-1},$$

etc., ... ; et soit

$$f(x_0, x_1, \ldots\ldots, x_{n-1};\ y_0,\ y_1, \ldots\ldots, y_{n-1};\ z_0,\ z_1, \ldots\ldots, z_{n-1};\ \ldots\ldots)$$

une fonction donnée de ces divers termes. Si à cette fonction l'on ajoute toutes celles que l'on peut en déduire, à l'aide d'un ou de plusieurs échanges opérés entre les lettres

$$x,\ y,\ z, \ldots\ldots$$

prises deux à deux, chacune des nouvelles fonctions étant prise avec le signe + ou avec le signe −, suivant qu'elle se déduit de la première par un nombre pair, ou par un nombre impair d'échanges; le résultat de cette addition sera une somme alternée par rapport aux suites dont il s'agit."

It is a little unfortunate that this definition proceeds on different lines from the others, being rather indeed a *rule for the formation* of an alternating function with respect to several sets of variables than a definition of such a function. It would have been much more appropriate and instructive to have said that a function was called *alternating with respect to two or more sets of the same number of variables* when the interchange of each member of a set with the corresponding member of another set altered the function in sign merely. Examples like the following could then have been given to make the two usages of the term perfectly clear, and to show the exact relation between them. To illustrate the first usage, the expressions

$$ac - bc,$$
$$(a-b)(c-d),$$
$$(a-b)(a-c)(b-c),$$

might be taken, where $ac - bc$ is an alternating function with respect to the variables a, b; $(a-b)(c-d)$ an alternating function with respect to a, b, and also with respect to c, d; and $(a-b)(a-c)(b-c)$ an alternating function with respect to a, b, with respect to a, c, and

with respect to b, c, or, shortly, an alternating function of all its variables. On the other hand, the expressions

$$a^2b - c^2d\,,$$
$$a\,b - c\,d\,,$$

would illustrate the second usage; $a^2b - c^2d$ being an alternating function with respect to the sets of variables ab, cd; and $ab - cd$ an alternating function with respect to the sets ab, cd, and also with respect to the sets ac, bd. In a word, the alteration which produces change of sign is, in the case of the first usage, interchange of two individual elements; in the case of the second usage it is interchange of two ranks or sets of elements.

The entity to which the new name *somme alternée* is given is explained as follows (p. 160):—

"Soit

$$f(x, y, z, \ldots\ldots)$$

une fonction quelconque de n variables

$$x, y, z, \ldots\ldots$$

et ajoutons à cette fonction toutes celles qu'on peut en déduire par la transposition des variables, ou, ce qui revient au même, par un ou plusieurs échanges opérés chacun entre deux variables seulement, chaque nouvelle fonction étant prise avec le signe + ou le signe −, suivant qu'elle se déduit de la première à l'aide d'un nombre pair ou impair de semblables échanges. La somme s ainsi obtenue sera la *somme alternée* que nous représentons par la notation

$$S\,[\pm f(x, y, z, \ldots)].$$

Ou trouvera, par exemple, en supposant $n = 2$,

$$s = f(x, y) - f(y, x)\,;$$

en supposant $n = 3$,

$$\begin{aligned} s = f(x,y,z) - f(x,z,y) &+ f(y,z,x) - f(y,x,z) \\ &+ f(z,x,y) - f(z,y,x), \end{aligned}$$

etc."

The only matter now remaining for explanation is the mode of transition from *sommes alternées* to *résultantes*, the difficult point

being, as in the memoir of 1812, to include all kinds of the latter as special cases of the former. The two pages which Cauchy devotes to the subject are curious to read, and deserve a little attention. He says (p. 161):—

> "Concevons maintenant que la fonction
>
> $$f(x,y,z,\ .\ .\ .\ )$$
>
> se reduise au produit de divers facteurs dont chacun renferme une suite des variables
>
> $$x,\ y,\ z,\ .\ .\ .\ .$$
>
> en sorte que l'on ait, par exemple,
>
> $$f(x,y,z,\ .\ .\ .\ ) = \phi(x)\chi(y)\psi(z)\ .\ .\ .\ .$$
>
> alors, pour obtenir la somme alternée
>
> $$s = S[\pm\phi(x)\chi(y)\psi(z)\ .\ .\ .\ ]$$
>
> il suffira . . ."

and having shown the mode of formation, and given the examples

$$s = \phi(x)\chi(y) - \phi(y)\chi(x),$$
$$s = \phi(x)\chi(y)\psi(z) - \phi(x)\chi(z)\psi(y)\ +\ .\ .\ .$$

he adds

> "Les sommes de cette espèce sont celles que M. Laplace a désignées sous le nom de *résultantes.*"

In regard to this the first comment clearly must be that it is not a little misleading. The sums referred to are only a very special class of those functions which Laplace called resultants; they belong, in fact, to that peculiar type for which in later times the name *alternant* was coined. In the second place, Cauchy's virtual renunciation of his own word "determinant" must be noted,—a renunciation all the more curious when we consider that the word had now been adopted by Jacobi, and had thereby become the recognised term in Germany. It may be that Laplace's word "resultant" had proved more acceptable in France, and that Cauchy merely bowed to the fact; but there is little or no evidence to support this.*

* Liouville, in a paper published in the same year as Cauchy's memoirs, uses *resultant*, but adds in a footnote, "Au lieu du mot *résultante*, les géomètres emploient souvent le mot *déterminant*" (*Liouville's Journ.*, vi. p. 348).

In the paragraph following the above Cauchy proceeds, as it were, to rectify matters. He says (p. 162):—

"Les formes des fonctions désignées par

$$\phi(x),\ \chi(x),\ \psi(x),\ \text{etc.}$$

étant arbitraires, aussi bien que les variables

$$x,\ y,\ z,\ \ldots,$$

permettent aux divers termes qui composent le tableau (2) d'acquérir des valeurs quelconques, et représentons ces variables à l'aide de lettres diverses

$$x,\ y,\ z,\ \ldots\ldots,\ t$$

affectés d'indices différents

$$0,\ 1,\ 2,\ \ldots,\ n-1,$$

dans les diverses lignes verticales. Alors, au lieu du tableau (2), on obtiendra le suivant

$$(5)\qquad \begin{cases} x_0, & x_1, & x_2, & \ldots\ldots, & x_{n-1} \\ y_0, & y_1, & y_2, & \ldots\ldots, & y_{n-1} \\ z_0, & z_1, & z_2, & \ldots\ldots, & z_{n-1} \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ t_0, & t_1, & t_2, & \ldots\ldots, & t_{n-1} \end{cases}$$

et la résultante s des termes dans ce dernier tableau sera

$$s = \mathrm{S}[\pm x_0 y_1 z_2 \ldots t_{n-1}]."$$

The general determinant is doubtless here reached, but the transition requisite for the attainment of it, viz., from $\phi(x)$, $\chi(x)$, $\psi(x)$, ... to the perfectly independent x_0, x_1, x_2, ... is not made without considerable strain. This is all the more surprising, too, when we consider, that a much less troublesome and less objectionable mode of bringing determinants under alternating aggregates lay ready to Cauchy's hand. Bearing in mind the definition given above, of *fonctions alternées par rapport à diverses suites*, we see that a determinant of the n^{th} order could have been made to appear as an alternating function with respect to n ranks of n variables each. For example, the determinant

$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2,$$

could have been introduced as a function alternating with respect to any two of the three ranks,

$$\begin{array}{ccc} a_1 & a_2 & a_3, \\ b_1 & b_2 & b_3, \\ c_1 & c_2 & c_3; \end{array}$$

and indeed, as we know, it is alternating also with respect to any two of the ranks

$$\begin{array}{ccc} a_1 & b_1 & c_1, \\ a_2 & b_2 & c_2, \\ a_3 & b_3 & c_3, \end{array}$$

that is to say, according to another phrase of Cauchy's, used above, it is alternating with respect to the indices 1, 2, 3.

The fourteen pages (pp. 163–176) which follow, are taken up with the properties of determinants as thus defined and with the application of them to the solution of simultaneous linear equations. Most of the matter is already familiar to us, and may be altogether passed over. One of the theorems it is necessary to give verbatim, not because of its importance, but because it serves to make evident the untenable position Cauchy had taken up in so peculiarly bringing determinants under the head of alternating aggregates. The theorem is (p. 164):—

> "Si, avec les variables comprises dans le tableau (5), on forme une fonction entière, du degré n, qui offre, dans chaque terme, n facteurs dont un seul appartienne à chacune des suites horizontales de ce tableau, et qui soit alternée par rapport à ces mêmes suites, la fonction entière dont il s'agit devra se réduire, au signe près, à la résultante s."

This not only justifies the definition proposed above to be substituted for Cauchy's, but it also entitles us to say that Cauchy having started by including determinants among alternating functions of one kind, viz., functions alternating with respect to every pair of n variables, soon succeeds in showing that they are alternating functions of an entirely different kind, viz., functions alternating with respect to every pair of n *ranks* of variables.

The only other noteworthy matter is a theorem in regard to

the solution of a set of simultaneous equations. Viewing the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= \xi \\ a_2x + b_2y + c_2z &= \eta \\ a_3x + b_3y + c_3z &= \zeta \end{aligned}\right\}$$

as giving each of the three variables ξ, η, ζ, in terms of the other three x, y, z, we see that on solving for x, y, z, we obtain a converse system, that is to say, a system giving each of the three x, y, z, in terms of ξ, η, ζ. The latter system is, as we know,

$$\left.\begin{aligned} x &= \frac{A_1}{\Delta}\xi + \frac{A_2}{\Delta}\eta + \frac{A_3}{\Delta}\zeta, \\ y &= \frac{B_1}{\Delta}\xi + \frac{B_2}{\Delta}\eta + \frac{B_3}{\Delta}\zeta, \\ z &= \frac{C_1}{\Delta}\xi + \frac{C_2}{\Delta}\eta + \frac{C_3}{\Delta}\zeta, \end{aligned}\right\}$$

where Δ is the determinant of the original system, and

$$A_1, B_1, C_1, A_2, \ldots,$$

are the cofactors in Δ of a_1, b_1, c_1, a_2, ... , respectively. Multiplying the determinants of the two systems, we obtain the determinant of the quantities

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{matrix}$$

Hence (p. 176):—

"Si, n variables

$$x, y, z, \ldots, t,$$

étant liées à n autres variables

$$\mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots, \mathrm{t},$$

par n équations linéaires, on suppose les unes exprimées en fonctions linéaires des autres, et réciproquement; les deux résultantes formées avec les coefficients que renfermeront ces fonctions linéaires dans les deux hypothèses, offriront un produit équivalent à l'unité." (XXI. 6)

Retrospect of the Period 1813–1841.

The characteristics of this period are best brought out by comparison with those of the preceding period, it being carefully borne in mind, in making the comparison, that the two are markedly unequal in length, the period of pioneering, as we may term it, extending to 120 years, and the next to only about 30.

In the first place, then, the evidence shows that as time went on there was considerable increase of interest in the subject, and a more widely spread knowledge of it; for, whereas to the longer period there belong 20 papers by 13 writers, for the shorter period the corresponding numbers are 35 and 18. Among the 18 writers, too, are represented nationalities which had previously not put in an appearance, viz., English, Italian, and Polish.

In the second place, we have proof that the early period was by far the more fruitful in original results. The pioneers had mapped out most of the prominent features of the new country; their successors had consequently to concern themselves in a considerable degree with filling in the details. During the second period one finds the fundamental propositions of the first period reproduced in new varieties of form; also, there are not awanting new proofs, extensions, and specialisations of old theorems; but of absolutely fresh departures there are comparatively few. An examination of the results numbered XLV.–LVIII. will show the character of these departures. It will be seen that they are due to Desnanot, Scherk, Schweins, Jacobi, Sylvester, and Cauchy. The most notable name of the period is Jacobi's, and next to it that of Schweins. There is no one name, however, which stands out in this period so conspicuously as Cauchy's does in the first period. Sylvester, unlike the others, it must be remembered, was only beginning his career, and we have yet to see him in the fulness of his power.

In the next place, the second period contrasts with the first in that during it important work was done on the subject of *special forms* of determinants. Here, again, the noteworthy names are those of Jacobi and Schweins.

Lastly, it having been noted in the retrospect of the first period that the subject of determinants was almost entirely a creation of the French intellect, we must not fail to take cognisance now of the fact that in the second period the pre-eminence belongs to Germany, France however taking still a fairly good second place.

ALPHABETICALLY ARRANGED LIST

OF

MATHEMATICIANS WHOSE WRITINGS ARE HEREIN REPORTED ON.

INDEX TO THE NUMBERED RESULTS.*

* This is merely temporary; it will be superseded in the next Part by a complete Index on a more elaborate plan.

ANNOUNCEMENT.

A New and Greatly Enlarged Edition of Dr Muir's "Treatise on the Theory of Determinants" is in course of preparation.